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RANDOM VIBRATION OF THIN ELASTIC PLATES AND SHALLOW SHELLS

**Hidekichi Kanematsu
William A. Nash**

July 1971

UNIVERSITY OF MASSACHUSETTS

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THIN ELASTIC PLATES AND SHALLOW SHELLS

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ABSTRACT

The large amplitude vibrations of thin elastic plates and shallow shells having various boundary conditions and subjected to random excitation are investigated by using various approximate techniques.

The random vibrations of rectangular plates and circular plates subjected to white random excitation are simulated numerically by two different methods. The first method is that the governing equations are reduced to a single-degree-of-freedom dynamical system and the reduced equation is then integrated numerically by the Runge-Kutta method employing the simulated approximate white noise as an input. The second method consists in integrating the equation of motion and the compatibility equation numerically by a finite-difference method employing the simulated approximate white noise as an input. To compare the results obtained by the simulation methods with those by other methods, the single-degree-of-freedom system equation is solved exactly using the Fokker-Planck equation, and solved approximately by the equivalent linearization technique. Also presented is the response analyses of shallow shells to white noise by (1) numerical simulation using the single-degree-of-freedom equation and (2) the Fokker-Planck equation. It has been shown that the solutions by the numerical simulation are close to those obtained by the equivalent linearization technique and the Fokker-Planck approach while the second numerical simulation gives rather poor solutions.

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ABSTRACT

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I INTRODUCTION

In many physical problem areas there are situations where a mechanical system is excited by a random type load and the response of the system displays a random trend. Since the response process is not deterministic, the response analysis must be treated statistically.

During the last two decades much research effort in the area of vibrations has been devoted to the investigation of the structural response to random excitation. Motivation for such research has arisen due to the development of large jet engines and rocket motors which produce random pressure fields of high intensity. Since the level of random excitation generated by jet aircraft and missiles provides a severe environment with respect to fatigue failure of structures, the investigation of the response of structures to random excitation plays an important role in the fields of aircraft and missile design.

The works of Crandall [1,2], Bolotin [3], Crandall and Mark [4], and Lin [5], contain various topics in random vibration analysis. So far the literature in the area of random vibration analysis. So far the literature in the area of random vibration has been surveyed by Crandall [6], Smith [7], Bolotin [8] and Vorovich [9]. Therefore in the present Chapter only random vibration of thin elastic plates and shallow shells will be reviewed briefly.

Analysis of the response of a linear elastic structure with known normal modes to random excitation of a given power spectrum have been carried out by many investigators [10 through 13]. Usually the input random load is assumed to be stationary, ergodic, Gaussian with zero mean value. For such

a case for a linear system it is possible to relate the statistical description of the output to that of the input. Hence, the mean-square response can be evaluated under the assumption of a stationary Gaussian input. However, the results of such analysis are valid only for small lateral displacements. For moderately large displacements it is necessary to take the effects of nonlinearities into account.

The motion of geometrically nonlinear elastic thin plates and shallow shells is described by a system of two coupled nonlinear partial differential equations in terms of the lateral displacement w and the stress function F , the so-called dynamic analog of the von Karman equations. The load is again considered to be a stationary Gaussian random process with zero mean value. Unfortunately, no exact solution for this problem has been found. Only approximate solutions are possible. One approximate solution is to reduce the partial differential equations to a system of ordinary differential equations for the generalized coordinates and to obtain an approximate solution by employing techniques used in nonlinear mechanics. To do this, assuming that the lateral displacement w is expressed as

$$w = \sum_{i=1}^N f_i(t) \psi_i(x, y) \quad (1-1)$$

where $f_i(t)$ are generalized coordinates and $\psi_i(x, y)$ is the coordinate function representing the deflected shape of the structure, and using approximation technique (for example Galerkin's method) the problem is reduced to a system of N dynamical equations with respect to $f_i(t)$, i.e.,

$$\frac{d^2 f_i(t)}{dt^2} + c \frac{df_i(t)}{dt} + g_i(f_1, f_2, \dots, f_n) = Q_i(t) \quad (1-2)$$

$i=1, 2, \dots, N$

where c = generalized damping term

g_i = generalized nonlinear stiffness term

Q_i = generalized random forcing function

An alternative form of the governing equations for $f_i(t)$ derived by mean of the Lagrangian equation takes the form

$$\frac{d^2 f_i(t)}{dt^2} + c \frac{df_i(t)}{dt} + \frac{\partial U}{\partial f_i} = Q_i(t) \quad (1-3)$$

$i = 1, 2, \dots, N$

where U is the potential energy of the system.

Only under the restriction that the generalized random forcing function $Q_i(t)$ is stationary, Gaussian, white noise or filtered white noise, can eqs. (1-2) or (1-3) be solved exactly in terms of the joint probability density using the Markov process and the associated Fokker-Planck equation [14]. This approach was first applied to the case of a nonlinear system by Andronov et al [15]. Herbert [16] investigated the multi-mode response of beams and plates to white noise excitation using this approach and showed that the probability density function of the model amplitude is non-Gaussian and statistically dependent. Dimentberg [23] applied the approach to the curved panel problem taking one term of the series for normal deflection in eq. (1-1) and studied the fatigue damage.

If the random forcing is not assumed to be white noise, only an approximate solution to eq. (1-2) or (1-3) is possible. One approximate technique for this type of equation is the equivalent linearization technique which was originated by Krylov-Bogoliubov [17] in deterministic theory and was applied to problems of random vibrations by Booton [18] and Caughey [19]. Lin [20] investigated the single-mode response of a flat, plate undergoing moderately large deflections subject to stationary Gaussian excitation whose power spectrum is relatively flat.

Another technique to obtain an approximate solution to eq.(1-2) is the perturbation method. This technique was first introduced to random vibration problems by Crandall [21]. This approach can be applied only to systems with very small nonlinearities.

Besides the above mentioned approximate approach, another method to estimate the response of a nonlinear elastic structure to random excitation is a numerical simulation technique which has been extensively used in the investigation of structural response due to earthquakes. This technique is to digitally simulate a physically realizable random load and to integrate the equation of motion numerically by employing the simulated random load as a forcing function. Then we compute the desired statistical properties of the response. Belz [22] used this technique to investigate the problem of a beam subjected to a concentrated random driving force. However, this method is time consuming if the structural model is complex. Instead of solving the original partial differential equation, eq.(1-2) may be integrated numerically.

In the present study, the large amplitude vibrations of plates and shallow shells having various boundary conditions and subjected to white random excitation will be investigated by using various approximate methods. Before going on to the analyses of the response to random excitation, an investigation of the structural response to a deterministic load will be made as a preliminary study in Chapter II.

In Chapter III, a Gaussian stationary white random process is digitally simulated.

In the second section of Chapter IV, the random vibrations of rectangular plates and circular plates subject to white noise are simulated numerically. Two different simulations are presented:

(1) the governing equations are reduced to a single-degree-of-freedom dynamical system by taking one term of the series representing the normal deflection in eq.(1-1) and the equation corresponding to the case of $N=1$ in eq. (1-2) is then integrated numerically by the Runge-Kutta method employing the simulated white noise as an input.

(2) the equation is integrated numerically by a finite-difference method employing the simulated white noise as an input. To compare the results obtained by the simulation method with those found by other methods, the single-degree-of freedom system equation is solved exactly using the Fokker-Planck equation. Also the approximate solutions obtained by the equivalent linearization technique are presented. In the third section of Chapter IV, the response analysis of shallow shells to white noise is carried out by (1) numerical simulation using the single-degree-of-freedom system equation and (2) the Fokker-Planck equation.

II

RESPONSE OF THIN ELASTIC PLATES AND
SHALLOW SHELLS TO STEP LOAD

In this Chapter, the response analysis of flat rectangular plates, flat circular plates, arbitrary shallow shells with rectangular boundaries and shallow spherical shells with a circular boundary to a uniformly distributed step load will be discussed. Nonlinear partial differential equations governing the finite amplitude deflections of plates and shallow shells are approximated by the finite-difference equations by use of the Crank-Nicolson finite-difference scheme [24] and these difference equations are then solved numerically using a CDC 3600 digital computer. The computer program for this analysis can be used for investigation of response to an arbitrary input forcing function if the input forcing function is digitally simulated. In Chapter IV the computer program written in this study will be used for digital simulation of random vibrations of plates. The purpose of the study in the present Chapter is that before simulating random vibrations we determine the numerical stability of the solution and the accuracy of the solution by investigating the response to a step load. Since the steady state response of the damped system to step load must agree with that under static load, we can check the computer program and the accuracy of the approximate solutions by comparing the values obtained here with existing results for the static load.

2.1 Rectangular Plates and Shallow Shells of Rectangular Contour2.1.1 Analysis

Governing Equations

Consider a thin, elastic, isotropic shallow shell rectangular in plan with double curvatures and of constant thickness h , Young's modulus E , and Poisson's ratio ν . The origin o of the curvilinear coordinates $x-y-z$ is chosen at a point of the middle surface corresponding to one of the corners of the shell. (See Figure 2-1) Let the oz axis extend along the normal to the middle surface toward the center of curvature. The ox and oy axes are drawn parallel to the lines of principal curvature of the shell. a and b denote the dimensions of the shell along the ox and oy axes. Also k_x and k_y are the curvatures of the shell which remain constant along the ox and the oy axes, respectively. Let w^* be the displacement of a point in the middle surface along the oz axis.

The differential equations governing the finite amplitude vibrations of such a shell are [25]

$$\begin{aligned} \rho h \frac{\partial^2 w^*}{\partial t^2} + c^* \rho h \frac{\partial w^*}{\partial t} + D \nabla^4 w^* &= h \frac{\partial^2 F^*}{\partial y^2} \left(k_x + \frac{\partial^2 w^*}{\partial x^2} \right) \\ + h \frac{\partial^2 F^*}{\partial x^2} \left(k_y + \frac{\partial^2 w^*}{\partial y^2} \right) - 2h \frac{\partial^2 F^*}{\partial x \partial y} \frac{\partial^2 w^*}{\partial x \partial y} + q^* & \end{aligned} \quad (2-1)$$

$$\nabla^4 F^* = E \left[\left(\frac{\partial^2 w^*}{\partial x \partial y} \right)^2 - \frac{\partial^2 w^*}{\partial x^2} \frac{\partial^2 w^*}{\partial y^2} - k_x \frac{\partial^2 w^*}{\partial y^2} - k_y \frac{\partial^2 w^*}{\partial x^2} \right] \quad (2-2)$$

where $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the shell, F^* is the Airy stress function, ρ is mass density of the material, c^* is the damping coefficient (assumed to be constant), q^* is the lateral load and t denotes time.

Membrane stresses σ_x^* , σ_y^* and τ_{xy}^* in the middle surface are given by

$$\begin{aligned}
 \sigma_x^* &= \frac{\partial^2 F^*}{\partial y^2} \\
 \sigma_y^* &= \frac{\partial^2 F^*}{\partial x^2} \\
 \tau_{xy}^* &= -\frac{\partial^2 F^*}{\partial x \partial y}
 \end{aligned} \tag{2-3}$$

The strains in the middle surface are expressed in terms of F^* as

$$\begin{aligned}
 \epsilon_x^* &= \frac{1}{E} \left(\frac{\partial^2 F^*}{\partial y^2} - \nu \frac{\partial^2 F^*}{\partial x^2} \right) \\
 \epsilon_y^* &= \frac{1}{E} \left(\frac{\partial^2 F^*}{\partial x^2} - \nu \frac{\partial^2 F^*}{\partial y^2} \right) \\
 \gamma_{xy}^* &= -\frac{2(1+\nu)}{E} \frac{\partial^2 F^*}{\partial x \partial y}
 \end{aligned} \tag{2-4}$$

Considering u^* and v^* which are the displacements in the middle surface in the x and y directions, respectively, the strains are expressed in terms of u^* , v^* and w^* as

$$\begin{aligned}
 \epsilon_x^* &= \frac{\partial u^*}{\partial x} - k_x w^* + \frac{1}{2} \left(\frac{\partial w^*}{\partial x} \right)^2 \\
 \epsilon_y^* &= \frac{\partial v^*}{\partial y} - k_y w^* + \frac{1}{2} \left(\frac{\partial w^*}{\partial y} \right)^2 \\
 \gamma_{xy}^* &= \frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x} + \frac{\partial w^*}{\partial x} \frac{\partial w^*}{\partial y}
 \end{aligned} \tag{2-5}$$

The bending moments are

$$M_x = -D \left(\frac{\partial^2 w^*}{\partial x^2} + \frac{\partial^2 w^*}{\partial y^2} \right) \tag{2-6}$$

$$M_y = -D \left(\frac{\partial^2 w^*}{\partial y^2} + \frac{\partial^2 w^*}{\partial x^2} \right)$$

(2-6)
cont'd

Non-dimensionalized Equations

Let us introduce the following dimensionless parameters:

$$\xi = \frac{x}{b}$$

$$\sigma_2 = \frac{\sigma_y}{E} \left(\frac{b}{h} \right)^2$$

$$K_0 = 12(1-v^2)$$

$$\tau_{xy}^* = \frac{\tau_{xy}}{E} \left(\frac{b}{h} \right)^2$$

$$\eta = \frac{y}{b}$$

$$u = \frac{u^* b}{h^2}$$

$$\tau = t \left(\frac{D}{\rho h b^4} \right)^{\frac{1}{2}}$$

$$v = \frac{v^* b}{h^2}$$

$$c = c^* \left(\frac{\rho h b^4}{D} \right)^{\frac{1}{2}}$$

$$k_1 = \frac{k_x a^2}{h}$$

$$w = \frac{w^*}{h}$$

$$k_2 = \frac{k_y b^2}{h}$$

$$F = \frac{F^*}{Eh^2}$$

$$q = q^* b^4 / Eh^4$$

$$\sigma_1 = \frac{\sigma_x^*}{E} \left(\frac{b}{h} \right)^2$$

$$\lambda = a/b = \text{aspect ratio}$$

(2-7)

Using the above dimensionless parameters, the equation of motion (2-1) and the compatibility equation (2-2) are now expressed by

$$\begin{aligned}
 \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} = -\left(\frac{\partial^4 F}{\partial \xi^4} + 2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4}\right) + K_0 \frac{\partial^2 F}{\partial \eta^2} \left(\frac{\partial^2 w}{\partial \xi^2} + \frac{k_1}{\lambda^2}\right) \\
 + K_0 \frac{\partial^2 F}{\partial \xi^2} \left(\frac{\partial^2 w}{\partial \eta^2} + k_2\right) - 2K_0 \frac{\partial^2 F}{\partial \xi \partial \eta} \frac{\partial^2 w}{\partial \xi \partial \eta} + K_0 q
 \end{aligned} \tag{2-8}$$

$$\frac{\partial^4 F}{\partial \xi^4} + 2 \frac{\partial^4 F}{\partial \xi^2 \partial \eta^2} + \frac{\partial^4 F}{\partial \eta^4} = \left(\frac{\partial^2 w}{\partial \xi \partial \eta}\right)^2 - \frac{\partial^2 w}{\partial \xi^2} \frac{\partial^2 w}{\partial \eta^2}$$

$$-k_1 \frac{\partial^2 w}{\partial \eta^2} - k_2 \frac{\partial^2 w}{\partial \xi^2} \tag{2-9}$$

Boundary Conditions and Initial Conditions

In this study, shallow rectangular shells with the following two boundary conditions are considered:

- a) All four edges are simply supported and immovable constrained against in-plane translation.
- b) All four edges are clamped and immovably constrained.

Let us formulate each set of boundary conditions. For the simply supported case, the deflection w along four edges must be zero and there is no bending moment along any edge. Thus, the analytical formulations of these boundary conditions in dimensionless form are from eq.(2-6),

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial \eta^2} + v \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{at } \eta = 0 \text{ and } \eta = 1$$

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial \xi^2} + v \frac{\partial^2 w}{\partial \eta^2} = 0 \quad \text{at } \xi = 0 \text{ and } \xi = \lambda$$

Since $w=0$ along $\eta=0$ and $\eta=1$, $\frac{\partial^2 w}{\partial \xi^2}$ must be zero. Also, $\frac{\partial^2 w}{\partial \eta^2}$ becomes zero along $\xi=0$ and $\xi=\lambda$. The above conditions can therefore be written as

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial \eta^2} = 0 \quad \text{at } \eta = 0 \text{ and } \eta = 1 \quad (2-10)$$

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial \xi^2} = 0 \quad \text{at } \xi = 0 \text{ and } \xi = \lambda$$

If the edges are also immovably constrained at the supports, the normal strain in the middle surface parallel to the edge must be zero along the edge. The boundary conditions are from eq.(2-4),

$$\frac{\partial^2 F}{\partial \xi^2} - \nu \frac{\partial^2 F}{\partial \eta^2} = 0 \quad \text{at } \xi = 0 \text{ and } \xi = \lambda \quad (2-11)$$

$$\frac{\partial^2 F}{\partial \eta^2} - \nu \frac{\partial^2 F}{\partial \xi^2} = 0 \quad \text{at } \eta = 0 \text{ and } \eta = 1$$

One additional condition required is that the relative displacement of the points on edges $\xi=0$ and $\xi=\lambda$ for any given value η is equal to zero.

$$u_{\xi=\lambda} - u_{\xi=0} = \int_0^\lambda \left(\frac{\partial u}{\partial \xi} \right)_{\eta=\text{constant}} d\xi = 0 \quad (2-12)$$

From eqs.(2-4) and (2-5), we have

$$\frac{\partial u}{\partial \xi} = \frac{\partial^2 F}{\partial \eta^2} - \nu \frac{\partial^2 F}{\partial \xi^2} - \frac{1}{2} \left(\frac{\partial w}{\partial \xi} \right)^2 + \frac{k_1}{\lambda^2} w$$

Substituting this expression into eq.(2-12) gives

$$\int_0^\lambda \left(\frac{\partial^2 F}{\partial \eta^2} - \nu \frac{\partial^2 F}{\partial \xi^2} \right) d\xi = \int_0^\lambda \left[\frac{1}{2} \left(\frac{\partial w}{\partial \xi} \right)^2 - \frac{k_1}{\lambda^2} w \right] d\xi \quad (2-13)$$

$\eta = \text{const.}$ $\eta = \text{const.}$

In a similar fashion, we have

$$\int_0^1 \left(\frac{\partial^2 F}{\partial \xi^2} - \nu \frac{\partial^2 F}{\partial \eta^2} \right) d\eta = \int_0^1 \left[\frac{1}{2} \left(\frac{\partial w}{\partial \eta} \right)^2 - k_2 w \right] d\eta \quad (2-14)$$

$\xi = \text{const.}$ $\xi = \text{const.}$

Next, consider the boundary conditions for clamped shells with immovable edges. In this case the deflections along the boundary are zero and the plane tangent to the deflected middle surface does not rotate at the edges.

Therefore, we have

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial \eta} = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \eta = 1 \quad (2-15)$$

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial \xi} = 0 \quad \text{at} \quad \xi = 0 \quad \text{and} \quad \xi = \lambda$$

The conditions (2-11), (2-13) and (2-14) must also be satisfied.

For initial conditions we assume the body to be at rest.

Then

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial \tau} = 0 \quad \text{at} \quad \tau = 0 \quad (2-16)$$

The transverse load applied to the shallow shell in this Chapter is a uniformly distributed step function type load which is expressed by

$$q(\xi, \eta, \tau) = Q u(\tau) \quad (2-17)$$

where

Q = amplitude of the load

$u(\tau)$ = unit step function defined by

$$u(\tau) = \begin{cases} 1 & \text{when } \tau > 0 \\ \frac{1}{2} & \text{when } \tau = 0 \\ 0 & \text{when } \tau < 0 \end{cases} \quad (2-18)$$

The problem consists in determining the functions w and F which satisfy eqs. (2-8) and (2-9) together with the prescribed boundary conditions and initial conditions. Because of difficulty in solving these simultaneous partial differential equations analytically, these equations will be solved numerically by a finite-difference method. The idea behind numerical integration of eq. (2-8) by the finite-difference method is the following: If the deflection w in the middle surface is specified at a certain time level τ_k , the stress function F and the membrane stresses are determined by the compatibility equation (2-9) and boundary conditions. Given membrane stresses at time level τ_k , we seek the deflection w at the next level $\tau_k + \Delta\tau$ with load Q under the assumption that the membrane stresses at each point remain constant in the time interval $[\tau_k, (\tau_k + \Delta\tau)]$. By this assumption, the equation of motion (2-8) is treated as a linear partial differential equation in the time interval $[\tau_k, (\tau_k + \Delta\tau)]$.

Equation (2-8) is reduced to two lower order differential equations by introducing the two variable W and V defined as follows:

$$W = \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} \quad (2-19)$$

$$V = \frac{\partial w}{\partial \tau} \quad (2-20)$$

If we differentiate eq. (2-19) with respect to τ and substitute eq.

(2-20), we have

$$\frac{\partial W}{\partial \tau} = \frac{\partial^2 V}{\partial \xi^2} + \frac{\partial^2 V}{\partial \eta^2} \quad (2-21)$$

which is the first desired equation. Substituting eqs. (2-19) and (2-20) into eq. (2-9) to get,

$$\begin{aligned} \frac{\partial V}{\partial \tau} + cV = & - \left(\frac{\partial^2 W}{\partial \xi^2} + \frac{\partial^2 W}{\partial \eta^2} \right) + K_0 \left(\sigma_1 \frac{\partial^2 w}{\partial \xi^2} + \sigma_2 \frac{\partial^2 w}{\partial \eta^2} + 2\tau_1 \frac{\partial^2 w}{\partial \xi \partial \eta} \right) \\ & + K_0 (\sigma_1 k_1 + \sigma_2 k_2 + Q) \end{aligned} \quad (2-22)$$

If we define the column matrix U as

$$U = \begin{pmatrix} W \\ V \end{pmatrix} \quad (2-23)$$

then, eqs.(2-21) and (2-22) can be written in matrix form as

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial U}{\partial \tau} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix} U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} \right) \\ + K_0 \begin{pmatrix} 0 \\ \sigma_1 \frac{\partial^2 W}{\partial \xi^2} + \sigma_2 \frac{\partial^2 W}{\partial \eta^2} + 2\tau_{12} \frac{\partial^2 W}{\partial \xi \partial \eta} + \sigma_1 k_1 + \sigma_2 k_2 + Q \end{pmatrix} \quad (2-24)$$

Formulation of the Finite-Difference Equation

If we restrict ourselves to the first mode type response of shells, it is only necessary to consider one-quarter of the shell because of the symmetry. Let us consider the rectangular network as shown in Figure 2-3 at time τ_k . The grid dimensions in the ξ and η directions and the time increment are denoted by $\Delta\xi$, $\Delta\eta$ and $\Delta\tau$, respectively. If one-quarter of the shell is divided into $(M-1) \times (N-1)$ sub-divisions, the spacing dimensions $\Delta\xi$ and $\Delta\eta$ are

$$\Delta\xi = \frac{\lambda}{2(M-1)}$$

$$\Delta\eta = \frac{1}{2(N-1)} \quad (2-25)$$

$$\gamma = \frac{\Delta\xi}{\Delta\eta} = \lambda \frac{(N-1)}{(M-1)}$$

Besides the interior points $(M-1) \times (N-1)$ and the boundary points $(M+N-1)$, fictitious points are introduced along the lines $\xi = -\Delta\xi$ and $\eta = -\Delta\eta$. Let us denote any variable Ψ at a discrete point $P_{i,j}$ and at time level τ_k to be $\Psi_{i,j}^k$. Hereafter, the subscript represents the position and the superscript denotes time level.

We shall reduce eq.(2-24) to the finite-difference form using Crank-Nicolson finite-difference scheme. The partial derivatives are approximated by

$$\begin{aligned}
 \left(\frac{\partial U}{\partial \tau}\right)_{i,j} &= (U_{i,j}^{k+1} - U_{i,j}^k)/\Delta\tau \\
 \left(\frac{\partial U}{\partial \xi}\right)_{i,j} &= (U_{i+1,j}^{k+1} - U_{i-1,j}^{k+1} + U_{i+1,j}^k - U_{i-1,j}^k)/4\Delta\xi \\
 \left(\frac{\partial^2 U}{\partial \xi^2}\right)_{i,j} &= (U_{i+1,j}^{k+1} - 2U_{i,j}^{k+1} + U_{i-1,j}^{k+1} + U_{i+1,j}^k - 2U_{i,j}^k \\
 &\quad + U_{i-1,j}^k)/2(\Delta\xi)^2
 \end{aligned} \tag{2-26}$$

$$(U)_{i,j} = (U_{i,j}^{k+1} + U_{i,j}^k)/2$$

and $(w)_{i,j}$ can be expressed in terms of $w_{i,j}^k$ and $U_{i,j}^k$ as follows:

$$(w)_{i,j} = (w_{i,j}^{k+1} + w_{i,j}^k)/2$$

$$v_{i,j}^k = (w_{i,j}^{k+1} - w_{i,j}^k)/\Delta\tau$$

From these two equations

$$(w)_{i,j} = w_{i,j}^k + \frac{\Delta\tau}{2}(0 \ 1) U_{i,j}^k \tag{2-27}$$

Using eqs. (2-26) and (2-27), eq.(2-24) is approximately written as

$$\begin{aligned}
 &\left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) \frac{U_{i,j}^{k+1} - U_{i,j}^k}{\Delta\tau} + \left(\begin{matrix} 0 & 0 \\ 0 & c/2 \end{matrix}\right) (U_{i,j}^{k+1} + U_{i,j}^k) = \left(\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}\right) \frac{1}{2(\Delta\tau)^2} \\
 &\times \left[\frac{1}{r^2} (U_{i+1,j}^{k+1} + U_{i-1,j}^{k+1} - 2U_{i,j}^{k+1}) + (U_{i,j+1}^{k+1} + U_{i,j-1}^{k+1} \right. \\
 &\quad \left. - 2U_{i,j}^{k+1}) + \frac{1}{r^2} (U_{i+1,j}^k + U_{i-1,j}^k - 2U_{i,j}^k) + (U_{i,j+1}^k + U_{i,j-1}^k \right. \\
 &\quad \left. - 2U_{i,j}^k) \right] + \left(\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}\right) \frac{K_o \sigma_i^k}{2(\Delta\eta)^2 r^2} (U_{i+1,j}^{k+1} + U_{i-1,j}^{k+1} - 2U_{i,j}^{k+1})
 \end{aligned}$$

$$\begin{aligned}
& + (0 \ 0) \frac{K_0^{\sigma_2}}{2(\Delta\eta)^2} \frac{k}{i,j} (U_{i,j+1}^{k+1} + U_{i,j-1}^{k+1} - 2U_{i,j}^{k+1}) + (0 \ 0) \frac{K_0^{\tau_{12}}}{4(\Delta\eta)^2 r^2} \frac{k}{i,j} \\
& \times (U_{i+1,j+1}^{k+1} - U_{i-1,j+1}^{k+1} - U_{i+1,j-1}^{k+1} + U_{i-1,j-1}^{k+1}) + (A)
\end{aligned} \tag{2-28}$$

where

$$\begin{aligned}
A = & K_0 [\sigma_1 \frac{k}{i,j} k_1 + \sigma_2 \frac{k}{i,j} k_2 + Q + \frac{\sigma_1}{(\Delta\eta)^2 r^2} \frac{k}{i,j} (w_{i+1,j}^k + w_{i-1,j}^k \\
& - 2w_{i,j}^k) + \frac{\sigma_2}{(\Delta\eta)^2} \frac{k}{i,j} (w_{i,j+1}^k + w_{i,j-1}^k - 2w_{i,j}^k) \\
& + \frac{\tau_{12}}{2(\Delta\eta)^2 r^2} \frac{k}{i,j} (w_{i+1,j+1}^k - w_{i-1,j+1}^k - w_{i+1,j-1}^k + w_{i-1,j-1}^k)]
\end{aligned} \tag{2-29}$$

Transposing the quantities at time τ_{k+1} to the left hand side and those at time τ_k to the right hand side, eq. (2-28) is written as

$$\begin{aligned}
& [D_9] U_{i+1,j+1}^{k+1} + [D_8] U_{i+1,j}^{k+1} + [D_7] U_{i+1,j-1}^{k+1} + [D_6] U_{i,j+1}^{k+1} \\
& + [D_5] U_{i,j}^{k+1} + [D_4] U_{i,j-1}^{k+1} + [D_3] U_{i-1,j+1}^{k+1} + [D_2] U_{i-1,j}^{k+1} \\
& + [D_1] U_{i-1,j-1}^{k+1} = [C_9] U_{i+1,j+1}^k + [C_8] U_{i+1,j}^k \\
& + [C_7] U_{i+1,j-1}^k + [C_6] U_{i,j+1}^k + [C_5] U_{i,j}^k + [C_4] U_{i,j-1}^k \\
& + [C_3] U_{i-1,j+1}^k + [C_2] U_{i-1,j}^k + [C_1] U_{i-1,j-1}^k + \begin{pmatrix} 0 \\ \Delta\tau A \end{pmatrix}
\end{aligned} \tag{2-30}$$

where $[D_m]$ and $[C_m]$ ($m=1, 2, \dots, 9$) are the square matrices defined by

$$[D_9] = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{K_0 \tau_{12}(\Delta\tau)^2}{4(\Delta\eta)^2 r} \end{bmatrix} \quad (2-31)$$

$$[D_8] = \begin{bmatrix} 0 & -\frac{\Delta\tau}{2(\Delta\eta)^2 r^2} \\ \frac{\Delta\tau}{2(\Delta\eta)^2 r^2} & -\frac{K_0 \sigma_1(\Delta\eta)^2}{2(\Delta\eta)^2 r^2} \end{bmatrix}$$

$$[D_7] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{K_0 \tau_{12}(\Delta\tau)^2}{4(\Delta\eta)^2 r} \end{bmatrix}$$

$$[D_6] = \begin{bmatrix} 0 & -\frac{\Delta\tau}{2(\Delta\eta)^2} \\ \frac{\Delta\tau}{2(\Delta\eta)^2} & -\frac{K_0 \sigma_2(\Delta\tau)^2}{2(\Delta\eta)^2} \end{bmatrix}$$

$$[D_5] = \begin{bmatrix} 1 & \frac{\Delta\tau}{(\Delta\eta)^2} \left(1 + \frac{1}{r^2}\right) \\ -\frac{\Delta\tau}{(\Delta\eta)^2} \left(1 + \frac{1}{r^2}\right) & 1 + \frac{K_0 \sigma_1(\Delta\eta)^2}{(\Delta\eta)^2 r^2} + \frac{K_0 \sigma_2(\Delta\eta)^2}{(\Delta\eta)^2} + \frac{\Delta\tau_c}{2} \end{bmatrix}$$

$$[D_4] = \begin{bmatrix} 0 & -\frac{\Delta\tau}{2(\Delta\eta)^2} \\ \frac{\Delta\tau}{2(\Delta\eta)^2} & -\frac{K_0 \sigma_2 (\Delta\tau)^2}{2(\Delta\eta)^2} \end{bmatrix} \quad (2-31)$$

cont'd

$$[D_3] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{K_0 \tau_{12} (\Delta\tau)^2}{4(\Delta\eta)^2 r} \end{bmatrix}$$

$$[D_2] = \begin{bmatrix} 0 & -\frac{\Delta\tau}{2(\Delta\eta)^2 r^2} \\ \frac{\Delta\tau}{2(\Delta\eta)^2 r^2} & -\frac{K_0 \sigma_1 (\Delta\tau)^2}{2(\Delta\eta)^2 r^2} \end{bmatrix}$$

$$[D_1] = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{K_0 \tau_{12} (\Delta\tau)^2}{4(\Delta\eta)^2 r} \end{bmatrix}$$

$$[C_8] = \begin{bmatrix} 0 & \frac{\Delta\tau}{2(\Delta\eta)^2 r^2} \\ -\frac{\Delta\tau}{2(\Delta\eta)^2 r^2} & 0 \end{bmatrix}$$

$$\begin{aligned}
 [C_5] &= \begin{bmatrix} 0 & -\frac{\Delta\tau}{(\Delta\eta)^2} \left(1 + \frac{1}{r^2}\right) \\ \frac{\Delta\tau}{(\Delta\eta)^2} \left(1 + \frac{1}{r^2}\right) & 1 - \frac{\Delta\tau c}{2} \end{bmatrix} \\
 [C_4] &= \begin{bmatrix} 0 & 2\frac{\Delta\tau}{(\Delta\eta)^2} \\ \frac{-\Delta\tau}{2(\Delta\eta)^2} & 0 \end{bmatrix} \tag{2-32}
 \end{aligned}$$

$$[C_6] = [C_8]$$

$$[C_2] = [C_8]$$

$[C_9]$, $[C_7]$, $[C_3]$ and $[C_1]$ are null matrices.

The compatibility equation (2-9) at time τ_k and at interior point $P_{i,j}$ ($3 \leq i \leq M+1$, $3 \leq j \leq N+1$) has the following finite-difference approximation:

$$\begin{aligned}
 &2F_{i,j}(3r^2 + 4 + \frac{3}{r^2}) + 2(F_{i+1,j+1} + F_{i-1,j+1} + F_{i-1,j-1} + F_{i+1,j-1}) \\
 &+ \frac{1}{r^2}F_{i+2,j} + r^2F_{i,j+2} + \frac{1}{r^2}F_{i-2,j} + r^2F_{i,j-2} - 4(1 + \frac{1}{r^2})F_{i+1,j} \\
 &- 4(1+r^2)F_{i,j+1} - 4(1 + \frac{1}{r^2})F_{i-1,j} - 4(r^2+1)F_{i,j-1} \\
 &= \frac{1}{16}(w_{i+1,j+1} - w_{i-1,j+1} + w_{i-1,j-1} - w_{i+1,j-1})^2 \\
 &- (w_{i+1,j} + w_{i-1,j} - 2w_{i,j}) (w_{i,j+1} + w_{i,j-1} - 2w_{i,j})
 \end{aligned}$$

(2-33)
cont'd

$$-k_1(w_{i,j+1} + w_{i,j-1} - 2w_{i,j}) (\Delta\eta)^2 - k_2(w_{i+1,j} + w_{i-1,j} - 2w_{i,j}) (\Delta\eta)^2$$

Now let us formulate the boundary conditions for the simply supported shells. For the point $P_{i,2}$ and $P_{2,m}$ on the boundary, eqs. (2-10) and (2-11) are written in the finite difference form as

$$w_{i,2} = 0 \text{ and } w_{i,3} = -w_{i,1} \quad 2 \leq i \leq M+1 \quad \text{at } \eta=0 \quad (2-34)$$

$$w_{2,m} = 0 \text{ and } w_{3,m} = -w_{1,m} \quad 2 \leq m \leq N+1 \quad \text{at } \xi=0$$

$$r^2(F_{i,3} + F_{i,1} - 2F_{i,2}) - v(F_{i+1,2} + F_{i-1,2} - 2F_{i,2}) = 0$$

$$2 \leq i \leq M+1 \quad \text{at } \eta=0$$

$$(F_{3,m} + F_{1,m} - 2F_{2,m}) - vr^2(F_{2,m+1} + F_{2,m-1} - 2F_{2,m}) = 0$$

$$2 \leq m \leq N+1 \quad \text{at } \xi = 0 \quad (2-35)$$

The integrals in eqs. (2-13) and (2-14) are approximated by the trapezoidal rule as follows. In the ξ direction, keeping η constant, eq. (2-13) is approximately expressed by

$$\begin{aligned} & \sum_{i=2}^{M-1} \left\{ \left(\frac{\partial^2 F}{\partial \eta^2} - v \frac{\partial^2 F}{\partial \xi^2} \right)_{i,j} + \left(\frac{\partial^2 F}{\partial \eta^2} - v \frac{\partial^2 F}{\partial \xi^2} \right)_{i+1,j} \right\} \\ & = \sum_{i=2}^{M-1} \left\{ \left[\frac{1}{2} \left(\frac{\partial w}{\partial \xi} \right)^2 - \frac{k_1}{\lambda^2} w \right]_{i,j} + \left[\frac{1}{2} \left(\frac{\partial w}{\partial \xi} \right)^2 - \frac{k_1}{\lambda^2} w \right]_{i+1,j} \right\} \\ & \quad 3 \leq j \leq N+1 \quad (2-36) \end{aligned}$$

Similarly, in the η direction, eq. (2-14) is expressed by

$$\sum_{j=2}^{N-1} \left\{ \left(\frac{\partial^2 F}{\partial \xi^2} - v \frac{\partial^2 F}{\partial \eta^2} \right)_{i,j} + \left(\frac{\partial^2 F}{\partial \xi^2} - v \frac{\partial^2 F}{\partial \eta^2} \right)_{i,j+1} \right\}$$

$$= \sum_{j=2}^{N-1} \left\{ \frac{1}{2} \left(\frac{\partial w}{\partial n} \right)_{i,j}^2 - k_2 w_{i,j} + \frac{1}{2} \left(\frac{\partial w}{\partial n} \right)_{i,j+1}^2 - k_2 w_{i,j+1} \right\}$$

$$3 \leq i \leq M+1 \quad (2-37)$$

Equations (2-36) and (2-37) are expressed by the following finite-difference equations:

$$\begin{aligned} & \sum_{i=2}^{M-2} \left\{ r^2 (F_{i,j+1} + F_{i,j-1} - 2F_{i,j}) - v(F_{i+1,j} + F_{i-1,j} \right. \\ & \quad \left. - 2F_{i,j}) + r (F_{i+1,j+1} + F_{i+1,j-1} - 2F_{i+1,j}) \right. \\ & \quad \left. - v(F_{i+2,j} + F_{i,j} - 2F_{i+1,j}) \right\} = \sum_{i=2}^{M-2} \left[\frac{1}{4} (w_{i+1,j} \right. \\ & \quad \left. - w_{i-1,j})^2 - (\Delta n)^2 k_1 w_{i,j} + \frac{1}{4} (w_{i+2,j} - w_{i,j})^2 \right. \\ & \quad \left. - (\Delta n)^2 k_1 w_{i+1,j} \right] \quad 3 \leq j \leq N+1 \quad (2-38) \end{aligned}$$

$$\begin{aligned} & \sum_{j=2}^{N-2} \left\{ \frac{1}{r^2} (F_{i+1,j} + F_{i-1,j} - 2F_{i,j}) - v(F_{i,j+1} + F_{i,j-1}) \right. \\ & \quad \left. - 2F_{i,j}) + \frac{1}{r^2} (F_{i+1,j+1} + F_{i-1,j-1} - 2F_{i,j+1}) \right. \\ & \quad \left. - v(F_{i,j+2} + F_{i,j} - 2F_{i,j+1}) \right\} \\ & = \sum_{j=2}^{N-2} \left[\frac{1}{4} (w_{i,j+1} - w_{i,j-1})^2 - (\Delta n)^2 k_2 w_{i,j} \right. \\ & \quad \left. + \frac{1}{4} (w_{i,j+2} - w_{i,j})^2 - (\Delta n)^2 k_2 w_{i,j+1} \right] \end{aligned}$$

$$3 \leq i \leq M+1 \quad (2-39)$$

For the clamped shells, the finite-difference forms of eq.(2-15) are

$$w_{i,2} = 0 \text{ and } w_{i,3} = w_{i,1} \quad 2 \leq i \leq M+1 \quad \text{at } \eta = 0 \quad (2-40)$$

$$w_{2,j} = 0 \text{ and } w_{3,j} = w_{1,j} \quad 2 \leq j \leq N+1 \quad \text{at } \xi = 0 \quad (2-41)$$

Equations (2-38) and (2-39) must hold for this case, too.

Procedure for Solving $w_{i,j}$ and $F_{i,j}$

Together with eqs.(2-33), (2-38) and (2-39), the following simultaneous equations for the stress function F at the interior points and the boundary points are formulated:

$$[A] \{F\} = \{G\} \quad (2-42)$$

where $[A]$ is the $(M+N+MN-1) \times (M+N+MN-1)$ square matrix whose elements are constants, $\{F\}$ is the column matrix of $F_{i,j}$ and $\{G\}$ is also the column matrix whose elements are functions of $w_{i,j}$. In the above formulation, the stress function at the corner $F_{2,2}$ has been assumed to be zero following Wang's assumption [26]. From eq.(2-42), if the deflection $w_{i,j}$ at each discrete point is given, then the stress function $F_{i,j}$ and the membrane stresses at each point can be determined.

The initial conditions (2-26) now take the form

$$w_{i,j}^0 = 0 \text{ and } U_{i,j}^0 = 0 \quad (2-43)$$

Equation (2-30) together with the boundary conditions (2-34) or (2-41) may be solved for $U_{i,j}^*$ using a digital computer by the following procedure.

Step.1 Given the initial conditions (2-43), eq. (2-42) is solved for $F_{i,j}^0$, from which the membrane stresses at each point $P_{i,j}$ are determined as

$$\sigma_{1i,j} = (F_{i,j+1} + F_{i,j-1} - 2F_{i,j}) / (\Delta\eta)^2 \quad (2-44)$$

$$\sigma_{2i,j} = (F_{i+1,j} + F_{i-1,j} - 2F_{i,j}) / r^2 (\Delta\eta)^2$$

$$\tau_{12i,j} = (F_{i+1,j+1} - F_{i-1,j+1} - F_{i+1,j-1} + F_{i-1,j-1}) / 4(\Delta\eta)^2 r \quad (2-45)$$

Step.2 Substituting these initial deflections, the membrane stresses and Q into eqs.(2-29), (2-31) and (2-32), eq.(2-30) represents the simultaneous equations for $U_{i,j}$. Solving for $U_{i,j}, w_{i,j}$ is then found from the relation

$$w_{i,j}^1 = w_{i,j}^0 + \Delta\tau (0, 1) U_{i,j}^1 \quad (2-46)$$

Step.3 Substitute $w_{i,j}^1$ into the right hand side of eq. (2-42) and solve for $F_{i,j}^1$. Calculate the membrane stresses by eq.(2-43). Return to step 2 and continue the procedure.

2.1.2 Examples

As an example of the present analysis, the steady state response of the following structures to various magnitudes of step load were determined.

1. Simply Supported Rectangular Plate (Aspect ratio $\lambda=1, 1.25, 1.5$ and 2.0)
2. Clamped Rectangular Plate (Aspect ratio $\lambda=1, 1.25, 1.5$ and 2.0)
3. Simply Supported Double Curvature Shell (Aspect ratio $\lambda=1$)
4. Clamped Double Curvature Shell (Aspect ratio $\lambda=1$) As previously mentioned, the supports do not permit in-plane displacement of the edges of the structure. For a rectangular plate with aspect ratio λ , the spacing dimension $\Delta\xi$ for fixing $N=4$ and $M=5$ was determined by (See Table 2-1)

$$\Delta\xi = \lambda \Delta\eta \frac{N-1}{M-1}$$

$$= \lambda/8 \quad (2-47)$$

If we choose M and $\Delta\xi$ such that

$$M = [1 + \lambda (N-1)] \quad (2-48)$$

$$\Delta\xi = \frac{\lambda}{2(M-1)} \quad (2-49)$$

the solutions will be more accurate than those obtained by eq.(2-47). However a much longer computing time is required. Therefore, the choice of $\Delta\xi$ was made using eq.(2-47) by sacrificing some accuracy of the solutions.

Since there is no criterion for determining the time increment $\Delta\tau$ to bring about numerical stability and convergence of the solution, $\Delta\tau$ was determined such that further decreasing $\Delta\tau$ did not appreciably affect the response. For the following values of $\Delta\tau$, the numerical solutions were apparently convergent:

Simply supported square plate $\Delta\tau \leq 5 \times 10^{-3}$

Clamped square plate $\Delta\tau \leq 5 \times 10^{-3}$

Simply supported rectangular plate with aspect ratio $\lambda=2$ $\Delta\tau \leq 7 \times 10^{-3}$

Clamped rectangular plate with aspect ratio $\lambda=2$ $\Delta\tau \leq 7 \times 10^{-3}$

For all cases, it was found that $\Delta\tau=0.002$ is sufficiently small. Through the present Chapter $\Delta\tau=0.002$ was used.

In Table 2-2 the results are listed together with Kornishin's static solutions [27] and the difference ϵ between the results obtained in the present study and his results are also listed for comparison. It can be seen that even if the interior points are 9 ($N=4$ and $M=4$) or 12 ($N=4$ and $M=5$) in number, reasonably good results were obtained. In Figure 2-5, a typical deflection response curve for the case of a clamped square plate is shown.

TABLE 2-1

CHOICE OF r , M AND N

λ	M	N	$r = \Delta\xi / \Delta n$
1	4	4	1
1.25	5	4	15/16
1.5	5	4	9/8
2	5	4	3/2

Table 2-2: Comparison of Steady State Response of Present Analysis with Static Analysis due to Kornishin

1. SIMPLY SUPPORTED RECTANGULAR PLATES

Aspect Ratio	Dimensionless Load	Present Study			Dimensionless Central Deflection	$\left \frac{w-w^*}{w^*} \right \times 100$
		Dimensionless Central Deflection	Dimensionless Membrane Stress	σ_1	σ_2	
λ	Q	w			w^*	ϵ
1.0	42.7	0.820	0.869	0.869	0.869	5.6
	96.1	1.126	1.230	1.230	1.230	8.5
	216.2	1.481	1.670	1.670	1.670	11.3
1.25	42.7	0.979	2.187	2.743	0.990	1.1
	96.1	1.290	4.106	4.990	1.380	6.5
	42.7	1.017	1.833	2.819	1.100	7.5
1.5	96.1	1.372	3.582	5.280	1.500	8.5
	42.7	1.040	1.361	2.821	1.135	8.4
2.0						

2. CLAMPED RECTANGULAR PLATES

Aspect Ratio	Dimensionless Load	Present Study			Kornishin Dimensionless Central Membrane Stress	$\left \frac{w-w'}{w'} \right \times 100$
		Dimensionless Central Deflection	Dimensionless Central Membrane Stress	Deflection		
λ	Q	w	σ_1	σ_2	w	ϵ
1.0	42.7	0.516	0.708	0.708	0.513	0.6
	96.1	0.909	2.105	2.105	0.909	0
	216.2	1.404	4.806	4.806	1.410	0.4
1.25	42.7	0.692	1.020	1.090	0.674	2.7
	96.1	1.120	2.570	2.700	1.110	1.0
	216.2	0.728	0.860	1.078	0.758	3.95
1.5	42.7	1.198	2.240	2.792	1.210	1.0
	96.1	1.767	6.685	9.395	1.750	1.0
	216.2	0.735	0.572	0.995	0.813	9.6
2.0	42.7	0.735	0.572	0.995	0.813	9.6

3. SIMPLY SUPPORTED DOUBLY CURVATURE SHELL

Aspect Ratio	Dimensionless Load	Curvature Parameter	Dimensionless Central Deflection		Dimensionless Central Membrane Stresses	
			k_1	k_2	w	σ_x
1.0	42.7	5	5	5	0.644	-1.358
1.5	20.0	5	5	5	0.367	-1.108
						-1.028

4. CLAMPED DOUBLY CURVATURE SHELL

Aspect Ratio	Dimensionless Load	Curvature Parameter	Dimensionless Central Deflection		Dimensionless Central Membrane Stresses	
			k_1	k_2	w	σ_x
1.0	42.7	5	5	5	1.642	0.560
1.5	20.0	5	5	5	1.750	-1.783
						0.765

2.2 Circular Plates and Shallow Spherical Shells

2.2.1 Analysis

Governing Equations

The finite amplitude vibration of a thin elastic, shallow spherical shell of thickness h , Young's modulus E , Poisson's ratio ν , constant curvature k^* and base radius R after assuming axisymmetry is governed by the following two simultaneous partial differential equations [25] in the polar coordinates system shown in Fig. 2-2.

$$\rho h \frac{\partial^2 w^*}{\partial t^2} + \rho h c^* \frac{\partial w^*}{\partial t} + D \nabla^4 w^* = h \frac{\partial F^*}{\partial r} \left(\frac{\partial^2 w^*}{\partial r^2} + k^* \right) + q^*(r, t) \quad (2-50)$$

$$\nabla^4 F^* = -E \frac{\partial^2 w^*}{\partial r^2} \frac{\partial w^*}{\partial r} - k^* E \left(\frac{\partial^2 w^*}{\partial r^2} + \frac{\partial w^*}{\partial r} \right) \quad (2-51)$$

where

$$\nabla^4 = \frac{\partial^4}{\partial r^4} + 2 \frac{\partial^3}{\partial r^3} - \frac{\partial^2}{\partial r^2 \partial r^2} + \frac{\partial}{\partial r^3 \partial r}$$

w^* = the normal displacement of a point in the middle surface

c^* = viscous damping

F^* = stress function

$q^*(r, t)$ = lateral load

D = $Eh^3/12(1-\nu^2)$

The membrane stresses in the middle surface are expressed by

$$\begin{aligned} \sigma_r^* &= \frac{\partial F^*}{\partial r} \\ \sigma_\theta^* &= \frac{\partial^2 F^*}{\partial r^2} \end{aligned} \quad (2-52)$$

The radial strain ϵ_r^* and the circumferential strain ϵ_θ^* are

$$\epsilon_r^* = \frac{\partial u^*}{\partial r} - k_w^* + \frac{1}{2} \left(\frac{\partial w^*}{\partial r} \right)^2 \quad (2-53)$$

$$\epsilon_\theta^* = \frac{u^*}{r} - k_w^*$$

where u^* is the radial displacement of a point in the middle surface of the shell.

The membrane stresses are also expressed in terms of ϵ_r^* and ϵ_θ^* as

$$\sigma_r^* = \frac{E}{1-\nu^2} (\epsilon_r^* + \nu \epsilon_\theta^*) \quad (2-54)$$

$$\sigma_\theta^* = \frac{E}{1-\nu^2} (\epsilon_\theta^* + \nu \epsilon_r^*)$$

From eqs. (2-52), (2-53) and (2-54)

$$\frac{u^*}{r} = k_w^* + \frac{1}{E} \left(\frac{\partial^2 F^*}{\partial r^2} - \nu \frac{\partial F^*}{r \partial r} \right) \quad (2-55)$$

$$\frac{\partial u^*}{\partial r} = k_w^* - \frac{1}{2} \left(\frac{\partial w^*}{\partial r} \right)^2 + \frac{1}{E} \left(\frac{\partial F^*}{r \partial r} - \nu \frac{\partial^2 F^*}{\partial r^2} \right)$$

Bending moments are given by

$$M_r^* = -D \left(\frac{\partial^2 w^*}{\partial r^2} + \frac{\nu}{r} \frac{\partial w^*}{\partial r} \right) \quad (2-56)$$

$$M_\theta^* = -D \left(\frac{\partial w^*}{r \partial r} + \frac{\partial^2 w^*}{\partial r^2} \right)$$

In the present study, the following two edge conditions will be considered.

- (a) Simply supported immovable edge
- (b) Clamped immovable edge

where again the edges cannot approach one another. Let us formulate these boundary conditions analytically.

(a) Simply supported shallow spherical shell with immovable edge

Since deflections w^* and the bending moment M_r^* are zero at the edge, from eq.(2-56) we have

$$w^* = 0 \text{ and } \frac{\partial^2 w^*}{\partial r^2} + \frac{\nu}{r} \frac{\partial w^*}{\partial r} = 0 \text{ at } r = R \quad (2-57)$$

Also the radial displacement u^* at the edge is zero.

From eq.(2-55),

$$\frac{u^*}{r} = k^* w^* + \frac{1}{E} \left(\frac{\partial^2 F^*}{\partial r^2} - \nu \frac{\partial F^*}{r \partial r} \right) = 0 \text{ at } r = R$$

But, since $w^* = 0$ at $r=R$,

$$\frac{\partial^2 F^*}{\partial r^2} - \nu \frac{\partial F^*}{r \partial r} = 0 \text{ at } r = R \quad (2-58)$$

Considering eq.(2-52), an additional restriction for F^* is that at the center of the shell, the membrane stress σ_r^* must be finite. Thus,

$$\frac{\partial F^*}{\partial r} = 0 \text{ at } r = 0 \quad (2-59)$$

Since $u^* = 0$ at $r=0$ and $r=R$, integration of the second equation of (2-55) from $r=0$ to $r=R$ leads to

$$\int_0^R \left\{ k^* w^* - \frac{1}{2} \left(\frac{\partial w^*}{\partial r} \right)^2 + \frac{1}{E} \left(\frac{\partial F^*}{r \partial r} - \nu \frac{\partial^2 F^*}{\partial r^2} \right) \right\} dr = 0 \quad (2-60)$$

which will be used for the formulation of the finite-difference equation.

(b) Clamped shallow spherical shell with immovable edge

The deflection and the slope at the edge are zero.

Thus

$$w^* = 0 \text{ and } \frac{\partial w^*}{\partial r} = 0 \text{ at } r=R \quad (2-61)$$

Conditions (2-58), (2-59) and (2-60) must hold for this case too.

For convenience, let us nondimensionalize all equations by introducing the following dimensionless parameters:

$$\begin{aligned}
w &= w^*/h & q &= q^* R^4/Eh^4 \\
s &= r/R & k &= k^* R^2/h \\
F &= F^*/Eh^2 & \sigma_s &= \sigma_r^* R^2/Eh^2 \\
\tau &= t(D/\rho h R^4)^{1/2} & \sigma_\theta &= \sigma_\theta^* R^2/Eh^2 \\
c &= c^* (\rho h R^4/D)^{1/2} & K_0 &= 12(1-\nu^2)
\end{aligned}
\tag{2-62}$$

Using eq.(2-62), eqs.(2-50) and (2-51) become

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} + \nabla^4 w &= K_0 \left[\frac{\partial^2 F}{\partial s^2} (k + \frac{\partial w}{s \partial s}) \right. \\
&\quad \left. + \frac{\partial F}{s \partial s} (k + \frac{\partial^2 w}{\partial s^2} + q) \right]
\end{aligned}
\tag{2-63}$$

$$\nabla^4 F = - \frac{\partial^2 w}{\partial s^2} \frac{\partial w}{s \partial s} - k \left(\frac{\partial^2 w}{\partial s^2} + \frac{\partial w}{s \partial s} \right)
\tag{2-64}$$

respectively. The boundary conditions (2-57) and (2-58) for simply supported edges are now expressed by

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial s^2} + \frac{\nu}{s} \frac{\partial w}{\partial s} = 0 \quad \text{at } s=1
\tag{2-65}$$

$$\frac{\partial^2 F}{\partial s^2} - \nu \frac{\partial F}{s \partial s} = 0 \quad \text{at } s=1
\tag{2-66}$$

respectively. Conditions (2-59) becomes

$$\frac{\partial F}{\partial s} = 0 \quad \text{at } s=0
\tag{2-67}$$

Equation (2-60) is transformed to

$$\int_0^1 \left(\frac{\partial F}{s \partial s} - \nu \frac{\partial^2 F}{\partial s^2} \right) ds = \int_0^1 \left[\frac{1}{2} \left(\frac{\partial w}{\partial s} \right)^2 - kw \right] ds
\tag{2-68}$$

The boundary conditions for the clamped edge (2-61) become

$$w = 0 \text{ and } \frac{\partial w}{\partial s} = 0 \quad \text{at } s=1 \quad (2-69)$$

The initial conditions imposed here are

$$w = 0 \text{ and } \frac{\partial w}{\partial \tau} = 0 \quad \text{at } \tau=0 \quad (2-70)$$

The problem for a simply supported edge consists in finding w and F from eqs. (2-63) and (2-64) subject to the boundary conditions (2-65) through (2-68) and initial conditions (2-70). For a clamped edge the boundary conditions are given by eq. (2-69) instead of eq. (2-65). The method of solution employed here is the same as that used for the rectangular shell.

To reduce the order of the derivatives in eq. (2-63), we introduce two new variable W and V defined by

$$W = \frac{\partial^2 w}{\partial s^2} \quad (2-71)$$

$$V = \frac{\partial w}{\partial \tau}$$

Eliminating w from above two equations, we have

$$\frac{\partial W}{\partial \tau} = \frac{\partial^2 V}{\partial s^2} \quad (2-72)$$

Using W and V , eq. (2-63) is expressed by

$$\begin{aligned} \frac{\partial V}{\partial \tau} + cV &= -\left(\frac{\partial^2 W}{\partial s^2} + \frac{2}{s} \frac{\partial W}{\partial s} - \frac{W}{s^2} + \frac{1}{s^3} \frac{\partial W}{\partial s}\right) \\ &\quad + K_0 \left[\sigma_s (W + 1) + \sigma_\theta \left(\frac{W}{s \partial s} + k \right) + q \right] \end{aligned} \quad (2-73)$$

If we use the vector U defined by

$$U = \begin{pmatrix} W \\ V \end{pmatrix} \quad (2-73)$$

eqs. (2-72) and (2-73) are expressed in the following matrix form:

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial U}{\partial \tau} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} U &= - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial^2 U}{\partial s^2} + \begin{bmatrix} 0 & 0 \\ \frac{2}{s} & 0 \end{bmatrix} \frac{\partial U}{\partial s} \\ &\quad + \begin{bmatrix} 0 & 0 \\ K_0 \left(\sigma_s + \frac{1}{s^2} \right) & 0 \end{bmatrix} U + \begin{bmatrix} 0 \\ K_0 q + (K_0 \sigma_\theta - \frac{1}{s^2}) \frac{\partial W}{s \partial s} + K_0 (\sigma_s + \sigma_\theta) k \end{bmatrix} \end{aligned} \quad (2-75)$$

Formulation of the Finite-Difference Equations

Consider a system of one dimensional equally spaced discrete points P_i^k at time τ_k as shown in Figure 2-4. Here the subscript i represents the spatial position and the superscript k denotes the time level. Because of the symmetry we consider the motion of only a radius of the shell. Let us divide the radius l into N intervals and call the point at the center of the shell P_0^k . One fictitious point P_{N+1}^k outside the edge is introduced. The spacing dimension Δs is l/N .

Let us write the finite-difference analog for eq.(2-75) using the Crank-Nicolson finite-difference scheme which was introduced in the previous section. Since we have a singularity at the center $s=0$, we must consider the case for $s=0$ separately.

At points P_i^k ($0 \leq i \leq N$), eq.(2-75) may be approximated by the following finite-difference expression:

$$\begin{aligned}
 & \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] \left\{ \frac{U_i^{k+1} - U_i^k}{\Delta \tau} \right\} + \left[\begin{array}{cc} 0 & 0 \\ -K_0 \sigma_{s_i}^k - \frac{1}{s_i^2} & c \end{array} \right] \left\{ \frac{U_i^{k+1} + U_i^k}{2} \right\} \\
 &= - \left[\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right] \left\{ \frac{U_{i+1}^{k+1} - U_{i-1}^{k+1} + U_{i+1}^k - U_{i-1}^k}{4 \Delta s} \right\} \\
 &+ \left[\begin{array}{c} 0 \\ K_0 q + K_0 k (\sigma_{\theta i}^k + \sigma_{s_i}^k) + (K_0 \frac{\sigma_{\theta i}^k}{s_i} - \frac{1}{s_i^3}) \left(\frac{w_{i+1}^k - w_{i-1}^k}{2 \Delta s} + \frac{\Delta \tau (v_{i+1}^{k+1} - v_{i-1}^{k+1})}{4 \Delta s} \right) \end{array} \right] \quad (2-76)
 \end{aligned}$$

Here it has been assumed that the membrane stresses remain constant in the time interval $[\tau_k, (\tau_k + \Delta \tau)]$. Rewriting eq. (2-76), we have

$$\begin{aligned}
 [B_3]U_{i+1}^{k+1} + [B_2]U_i^{k+1} + [B_1]U_{i-1}^{k+1} \\
 = [A_3]U_{i+1}^k + [A_2]U_i^k + [A_1]U_{i-1}^k + \begin{bmatrix} 0 \\ G_i^k \end{bmatrix}
 \end{aligned} \tag{2-77}$$

where $[B_j]$ and $[A_j]$ ($j=1, 2, 3$) are square metrices defined by

$$[B_3] = \begin{bmatrix} 0 & -\frac{\Delta\tau}{2(\Delta s)^2} \\ \frac{\Delta\tau}{2\Delta s s_i} & \frac{(\Delta\tau)^2}{4\Delta s s_i} \left(\frac{1}{s_i^2} - K_0 \sigma_{\theta i}^k \right) \end{bmatrix} \tag{2-78}$$

$$[B_2] = \begin{bmatrix} 1 & \frac{\Delta\tau}{(\Delta s)^2} \\ \frac{\Delta\tau}{2} \left(K_0 \sigma_{s i}^k + \frac{1}{s_i^2} + \frac{1}{(\Delta s)^2} \right) & 1 + \frac{c}{2} \Delta\tau \end{bmatrix}$$

$$[B_1] = \begin{bmatrix} 0 & -\frac{\Delta\tau}{2(\Delta s)^2} \\ \frac{\Delta}{2\Delta s} \left(\frac{1}{\Delta s} - \frac{1}{s_i} \right) & -\frac{(\Delta\tau)^2}{4\Delta s s_i} \left(\frac{1}{s_i} - K_0 \sigma_{\theta i}^k \right) \end{bmatrix}$$

$$[A_3] = \begin{bmatrix} 0 & \frac{\Delta\tau}{2(\Delta s)^2} \\ -\frac{\Delta\tau}{2\Delta s} \left(\frac{1}{\Delta s} + \frac{1}{s_i} \right) & 0 \end{bmatrix}$$

$$[A_2] = \begin{bmatrix} 1 & \frac{\Delta\tau}{(\Delta s)^2} \\ \frac{\Delta\tau}{2} \left(K_0 \sigma_{s i}^k + \frac{1}{s_i^2} + \frac{1}{(\Delta s)^2} \right) & 1 - \frac{c\Delta\tau}{2} \end{bmatrix}$$

$$[A_1] = \begin{bmatrix} 0 & \frac{\Delta\tau}{2(\Delta s)^2} \\ \frac{\Delta\tau}{2\Delta s} \left(\frac{1}{s_i} - \frac{1}{\Delta s} \right) & 0 \end{bmatrix}$$

and G_i^k is defined by

$$G_i^k = K_0 \Delta \tau [q + K(\sigma_{\theta i}^k + \sigma_{s i}^k)] + \Delta \tau \left(\frac{\sigma_{\theta i}^k}{s_i} - \frac{1}{s_i^3} \right) \left\{ \frac{w_{i+1}^k - w_{i-1}^k}{2\Delta s} \right\}$$

The compatibility equation (2-64) at the point P_i^k ($1 \leq i \leq N-1$) may be written in the finite-difference form

$$\begin{aligned} & (F_{i+2} - 4F_{i+1} + 6F_i - 4F_{i-1} + F_{i-2})/(\Delta s)^2 + (F_{i+2} - 2F_{i+1} + 2F_{i-1} \\ & - F_{i-2})/s_i(\Delta s)^3 - (F_{i+1} - 2F_i + F_{i-1})/(s_i \Delta s)^2 + (F_{i+1} - F_{i-1}) \\ & \times (1/2\Delta s s_i^3) = - (w_{i+1} - 2w_i + w_{i-1})(w_{i+1} - w_{i-1})/2s_i(\Delta s)^3 \\ & - k(w_{i+1} - 2w_i + w_{i-1})/(\Delta s)^2 - k(w_{i+1} - w_{i-1})/2s_i(\Delta s) \end{aligned} \quad (2-79)$$

At the point P_0^k , i.e., the center of the shell, using L'Hospital's rule, eq.(2-75) takes the form

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial U}{\partial \tau} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} U = - \begin{bmatrix} 0 & -1 \\ \frac{8}{3} & 0 \end{bmatrix} \frac{\partial^2 U}{\partial s^2} + K_0 (\sigma_s + \sigma_\theta) \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} U \\ + \begin{bmatrix} 0 \\ K_0 (q + \sigma_s + \sigma_\theta) \end{bmatrix} \quad (2-80)$$

which has the following finite-difference analog:

$$\begin{aligned} & 2[B_3^k]U_1^{k+1} + [B_2^k]U_0^{k+1} = 2[A_3^k]U_1^k + [A_2^k]U_0^k \\ & + \begin{bmatrix} 0 \\ K_0 [q + k(\sigma_{s0}^k + \sigma_{\theta0}^k)] \Delta \tau \end{bmatrix} \quad (2-81) \end{aligned}$$

where

$$[B_3^1] = \begin{bmatrix} 0 & -\frac{\Delta\tau}{2(\Delta s)^2} \\ \frac{4\Delta}{3(\Delta s)^2} & 0 \end{bmatrix}$$

$$[B_2^1] = \begin{bmatrix} 1 & \frac{\Delta\tau}{(\Delta s)^2} \\ \frac{1}{2}k_0(\sigma_{\theta 0}^k + \sigma_{s 0}^k)\Delta\tau - \frac{8}{3} \frac{\Delta\tau}{(\Delta s)^2} & \frac{1}{2}c\Delta\tau + 1 \end{bmatrix} \quad (2-82)$$

At $s=0$, the compatibility equation (2-64) can be expressed as (See APPENDIX),

$$\frac{8\partial^4 F}{3\partial s^4} = -\left(\frac{\partial^2 w}{\partial s^2}\right)^2 - 2k\frac{\partial^2 w}{\partial s^2} \quad (2-83)$$

which reduces to

$$\frac{4}{3}(F_2 - 4F_1 + 3F_0) = - (w_1 - w_0) - k(w_1 - w_0) (\Delta s)^2 \quad (2-84)$$

The boundary conditions (2-65) and (2-66) are expressed by

$$w_N = 0 \text{ and } \frac{w_{N+1} + w_{N-1}}{(\Delta s)^2} + \frac{v}{s_N} \frac{w_{N+1} - w_{N-1}}{2\Delta s} = 0 \quad (2-85)$$

$$\frac{F_{N+1} + F_{N-1} - 2F_N}{(\Delta s)^2} - \frac{F_{N+1} - F_{N-1}}{2s_N \Delta s} = 0 \quad (2-86)$$

respectively. The integral in eq.(2-68) is approximated by the trapezoidal rule as

$$\begin{aligned} & \frac{1}{2}(1-v) \left(\frac{\partial^2 F}{\partial s^2}\right)_0 + \sum_{i=1}^{N-1} \left(\frac{\partial F}{\partial s} - \frac{\partial^2 F}{\partial s^2} \right)_i + \frac{1}{2} \left(\frac{\partial F}{\partial s} - v \frac{\partial^2 F}{\partial s^2} \right)_N \\ &= \frac{1}{2} \left[\frac{1}{2} \left(\frac{\partial w}{\partial s} \right)_0^2 - kw_0 \right] + \sum_{i=1}^{N-1} \left[\frac{1}{2} \left(\frac{\partial w}{\partial s} \right)_i^2 - kw_i \right] + \frac{1}{2} \left[\frac{1}{2} \left(\frac{\partial w}{\partial s} \right)_N^2 - kw_N \right] \end{aligned} \quad (2-87)$$

In finite-difference form, this becomes

$$\begin{aligned} & (1-v) \frac{F_1 + F_0}{(\Delta s)^2} + \sum_{i=1}^{N-1} \frac{F_{i+1} - F_{i-1}}{2s_i \Delta s} - \frac{F_{i+1} + F_{i-1} - 2F_i}{(\Delta s)^2} \\ &+ \frac{1}{2} \left(\frac{F_{N+1} - F_{N-1}}{2s_N \Delta s} - \frac{F_{N+1} + F_{N-1} - 2F_N}{(\Delta s)^2} \right) \end{aligned}$$

$$\begin{aligned}
 &= -\frac{1}{2}kw_0 + \sum_{i=1}^{N-1} \left(\frac{(w_{i+1} - w_{i-1})^2}{8(\Delta s)^2} - kw_i \right) \\
 &\quad + \frac{1}{2} \left(\frac{1}{8} \frac{(w_{N+1} - w_{N-1})^2}{(\Delta s)^2} - kw_N \right)
 \end{aligned} \tag{2-88}$$

The boundary conditions (2-69) for a clamped edge are expressed by

$$w_N = 0 \text{ and } w_{N+1} = w_{N-1} \tag{2-89}$$

Also, conditions (2-86) and (2-88) must be satisfied for clamped shells.

The initial conditions (2-70) have the form

$$w_i^0 = 0 \text{ and } v_i^0 = 0 \quad i=0,1,\dots,N+1 \tag{2-90}$$

Procedure for solving for w_i and F_i

Equations (2-79), (2-84) and (2-89) constitute the following simultaneous equations for F_i :

$$[K]\{F\} = \{L\} \tag{2-91}$$

where $[K]$ is an $(N+1) \times (N+1)$ matrix whose elements are function of Δs , and s_i , $\{F\}$ is a column matrix of F_i , and $\{L\}$ is a column matrix whose elements are function of w_i^k . If the deflections at every point are known, then the stress function F at every point can be determined by solving eq.(2-91), namely

$$\{F\} = [K]^{-1} \{L\} \tag{2-92}$$

The procedure for solving for w_i is essentially the same as that described in the previous section (2-1-1):

Step 1. Substitute the initial conditions (2-90) into $\{L\}$ of eq.(2-92) and obtain F_i^0 . Then the membrane stresses at each point can be calculated by

$$\begin{aligned}\sigma_{s_i} &= \frac{F_{i+1} - F_{i-1}}{2s_i \Delta s} & 1 \leq i \leq N \\ \sigma_{\theta_i} &= \frac{F_{i+1} + F_{i-1} - 2F_i}{(\Delta s)^2}\end{aligned}\quad (2-93)$$

At the center of the shell $s=0$, using L'Hospital's rule

$$\sigma_{s_0} = \sigma_{\theta_0} = 2(F_1 - F_0)/(\Delta s)^2 \quad (2-94)$$

Step 2. Substitute σ_{s_i} and σ_{θ_i} into (2-78) and (2-82), and compute U_i^0 . Solve eqs. (2-77) and (2-81) for U_i^1 . The deflection w_i^1 can be obtained from

$$w_i^1 = w_i^0 + \Delta\tau(0,1)U_i^1$$

$$= w_i^0 + \Delta\tau V_i^1$$

Step 3. Substitute the new deflection w_i^1 into {L} of eq. (2-92) and compute F_i^1 . Repeat this procedure.

2.2.2 Examples

As an example, the computation of the deflection response was carried out for the following cases.

- 1) Simply Supported Circular Plate
- 2) Clamped Circular Plate
- 3) Simply Supported Spherical Shell ($k=1, 2$, and 3)
- 4) Clamped Spherical Shell ($k=1, 2$, and 3)

where the edges cannot approach one another. The spacing Δs was taken to be 0.2. The time increment $\Delta\tau$ for numerical stability and convergence of the solution for the given $\Delta s=0.2$ was found to be

$\Delta\tau \leq 1.0 \times 10^{-2}$ for a simply supported edge

$\Delta\tau \leq 2.5 \times 10^{-3}$ for a clamped edge

However, throughout this section the deflection response was calculated using $\Delta\tau = 0.002$. Typical deflection response curves for a clamped spherical shell are shown in Fig.2-6. The steady state responses of the above cases in the presence of heavy damping are listed in Table 2-3 together with Kornishin's static solutions.

Table 2-3: Comparison of Steady State Response of Present Analysis with Static Analysis due to Kornishin

1. Simply Supported Circular Plate

Dimensionless Load	Dimensionless Central Deflection		Dimensionless Central Membrane Stresses		$\left \frac{w-w^*}{w^*} \right \times 100$	$\left \frac{\sigma_s - \sigma_s^*}{\sigma_s^*} \right \times 100$
	Present Study	Kornishin	Present Study	Kornishin		
Q	w	w*	$\sigma_s = \sigma_\theta^*$	$\sigma_s^* = \sigma_\theta^*$		
10	1.34	1.38	1.86	1.76	2.9	5.7

2. Clamped Circular Plate

Dimensionless Load	Dimensionless Central Deflection		Dimensionless Central Membrane Stresses		$\left \frac{w-w^*}{w^*} \right \times 100$	$\left \frac{\sigma_s - \sigma_s^*}{\sigma_s^*} \right \times 100$
	Present Study	Kornishin	Present Study	Kornishin		
Q	w	w*	$\sigma_s = \sigma_\theta^*$	$\sigma_s^* = \sigma_\theta^*$		
10	1.08	1.06	1.11	1.04	2.4	6.7
15	1.33	1.32	1.66	1.60	0.8	3.8

3. SIMPLY SUPPORTED SHALLOW SPHERICAL SHELL

Curvature Parameter K	Dimensionless Load Q	Dimensionless Central Deflection		$\left \frac{w-w^*}{w^*} \right \times 100$	
		Present Study			
		w	w^*		
1	10	1.890	1.91	1.0	
2	1	0.142	0.16	11.2	
3	3	0.206	0.24	14.2	

4. CLAMPED SHALLOW SPHERICAL SHELL

Curvature Parameter K	Dimensionless Load Q	Dimensionless Central Deflection		$\left \frac{w-w^*}{w^*} \right \times 100$	
		Present Study			
		w	w^*		
1	10	0.938	0.86	9.0	
2	10	0.925	0.84	10.0	
3	10	0.352	0.40	12.0	

III

SIMULATION OF A STATIONARY GAUSSIAN WHITE NOISE

3.1 White Noise

White noise is a mathematical idealization of a stationary random process in which the power spectral density is constant, G_0 at all frequencies. Since the autocorrelation function $R(\tau)$ for white noise is expressed by

$$R(\tau) = 2\pi G_0 \delta(\tau) \quad (3-1)$$

where $\delta(\tau)$ is the Dirac delta function, the process has an infinite variance and is completely uncorrelated at different times.

3.2 Simulation of Approximate White Noise Processes

True white noise is physically impossible and cannot be simulated because it requires an infinite mean square value. Therefore in this section, approximate Gaussian white noise whose power spectral density is constant over the range of frequencies of interest and falls to zero as the frequency tends to infinity is generated. To do this, first a sequence of independent random numbers B_k distributed uniformly in the interval $(0,1)$ is generated by a power residue method [29] on the CDC 3600 digital computer. Then a white sequence of Gaussian numbers v_k with mean zero and variance σ_v^2 is obtained through the transformations

$$v_k = \sigma_v (-2 \log_e B_k)^{1/2} \left. \begin{array}{l} \cos 2\pi B_{k+1} \\ \sin 2\pi B_{k+1} \end{array} \right\} \quad k \text{ odd} \quad (3-2)$$

$$v_{k+1} = \sigma_v (-2 \log_e B_k)^{1/2}$$

Construct a function $p(t)$ consisting of a sequence of step functions having constant time interval Δh , with the ordinates in the various intervals being white numbers v_k . The mathematical formulation of the function $p(t)$ is

$$p(t) = \sum_{k=-\infty}^{\infty} v_k \{u[t - (k+1)\Delta h] - u(t - k\Delta h)\} \quad (3-3)$$

where $u(t)$ is a unit step function defined by

$$u(t) = \begin{cases} 1 & t > 0 \\ 1/2 & t = 0 \\ 0 & t < 0 \end{cases} \quad (3-4)$$

In this study, the process is assumed to be ergodic, stationary, and Gaussian. Hence ensemble average may be replaced by time average.

The temporal mean value μ_p and variance σ_p^2 become, respectively,

$$\mu_p = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T p(t) dt \quad (3-5)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{k=-N}^N v_k = 0$$

$$\sigma_p^2 = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T [p(t)]^2 dt \quad (3-6)$$

$$= \lim_{N \rightarrow \infty} \frac{1}{2N} \sum_{k=-N}^N v_k^2 = \sigma_r^2$$

The autocorrelation function of the random process $p(t)$ is calculated from

$$R(\tau) = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T p(t)p(t+\tau) dt \quad (3-7)$$

which yields

$$R(\tau) = \begin{cases} \sigma_v^2 \left(1 - \frac{|\tau|}{\Delta h}\right) & |\tau| \leq \Delta h \\ 0 & |\tau| \geq \Delta h \end{cases} \quad (3-8)$$

The power spectral density function $G(\omega)$ is the Fourier transform of the autocorrelation function $R(\tau)$. Since $R(\tau)$ is an even function of τ , then we have

$$\begin{aligned} G(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-\omega i \tau} d\tau \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} R(\tau) \cos \omega \tau d\tau \\ &= \frac{\sigma_v^2 \Delta h}{\pi} \left(\frac{\sin \frac{\omega \Delta h}{2}}{\frac{\omega \Delta h}{2}} \right) \end{aligned} \quad (3-9)$$

For $\omega \Delta h$ small, $G(\omega)$ can be approximated by

$$G(\omega) \approx \frac{\sigma_v^2 \Delta h}{\pi} \left[1 - \frac{(\omega \Delta h)^2}{12} \right] \quad (3-10)$$

Choosing Δh sufficiently small $G(\omega)$ can be approximated by $\sigma_v^2 \Delta h / \pi$.

IV

STATIONARY RESPONSE OF PLATES AND SHALLOW
SHELLS TO GAUSSIAN WHITE RANDOM EXCITATION4.1 Introduction

In the present Chapter the response analysis of plates and shallow shells to stationary Gaussian random excitation will be carried out by various methods. All equations and nomenclatures used here are the same as those described in Chapter II.

The random excitation $q(\xi, n, \tau)$ or $q(s, \tau)$ is assumed to be uniformly distributed over the structure and applied normal to the middle surface of the structure. Furthermore, the random excitation is also assumed to be stationary, ergodic and Gaussian white noise with zero mean value, that is

$$\begin{aligned} q(\xi, n, \tau) \text{ or } q(s, \tau) &= Q(\tau) \\ E[Q(\tau)] &= 0 \end{aligned} \tag{4-1}$$

$$E[Q(\tau_1)Q(\tau_2)] = 2\pi G_0 \delta(\tau_1 - \tau_2)$$

where G_0 is constant, $\delta(\tau)$ is the Dirac delta function and $E[]$ is the expectation operator. The methods of solution are the following.

Method I. Numerical Simulation by the Runge-Kutta Method

The equation of motion and the compatibility equation are reduced to a single-degree-of-freedom dynamical system. To do this we assume the normal deflection $w(\xi, n, \tau)$ or $w(s, \tau)$ to be the product of the coordinate function $\psi(\xi, n)$ which satisfies the boundary conditions and is suitable for representing the deflected shape of plates or shells and the generalized coordinate $f(\tau)$ such as

$$\begin{aligned} w(\xi, \eta, \tau) &= f(\tau) \psi(\xi, \eta) \\ w(s, \tau) &= f(\tau) \psi(s) \end{aligned} \quad (4-2)$$

Then we substitute eq. (4-2) into the compatibility equation and solve for the stress function F in terms of $f(\tau)$. Upon using Galerkin's method [25], the equation of motion and compatibility equation yield

$$\ddot{f} + 2\zeta\omega_0 \dot{f} + \omega_0^2(f + \beta f^2 + \gamma f^3) = \alpha Q(\tau) \quad (4-3)$$

where ζ is the fraction of critical damping. ω_0^2 , β , γ and α are constants. Note that for a flat plate β is zero.

The approximate white noise process $p(\tau)$ is generated digitally by following the method described in Chapter III, by use of eqs. (3-2) and (3-3). The power spectral density is expressed approximately by

$$G(\omega) \approx \frac{\sigma_v^2 \Delta h}{\pi} \left[1 - \frac{(\omega \Delta h)^2}{12} \right] \quad (4-4)$$

If Δh is small and the damping of the system ζ is small, $G(\omega)$ is almost flat near the frequency ω_0 and $p(\tau)$ can be considered to be white noise. Therefore,

$$G(\omega) \approx \sigma_v^2 \Delta h / \pi = G_0 \quad (4-5)$$

The magnitude of the power spectral density can be varied by changing either σ_v or Δh . Equation (4-3) is now integrated numerically by the Runge-Kutta method employing the $p(\tau)$ from eq. (3-4) as an input $Q(\tau)$. In the present study, the integration time increment $\Delta \tau$ and the basic time increment Δh are chosen as

$$\Delta \tau = \Delta h = 1/10\omega_0 \quad (4-6)$$

Since the random process is assumed to be ergodic, the ensemble average may be replaced by the temporal average. Therefore the mean, mean-square, and the variance of the response $f(\tau)$ are computed by

$$\begin{aligned}
 E[f] &\approx \frac{1}{T} \int_0^T f \, d\tau \\
 &\approx \frac{1}{N} \sum_{k=0}^N f_k \\
 E[f^2] &\approx \frac{1}{N} \sum_{k=0}^N f_k^2 \\
 \sigma_f^2 &\approx \frac{1}{N} \sum_{k=0}^N f_k^2 - \left(\frac{1}{N} \sum_{k=0}^N f_k \right)^2
 \end{aligned} \tag{4-7}$$

where σ_f^2 is the variance of $f(\tau)$ and f_k is the response of f at time τ_k . The mean-square, $E[w^2]$, and the variance, σ_w^2 , of the central deflection are found from eqs. (4-2) and (4-7) as

$$\begin{aligned}
 E[w^2] &= E[f^2] \psi^2 (1/2\lambda, 1/2) \\
 \sigma_w^2 &= \sigma_f^2 \psi^2 (1/2\lambda, 1/2)
 \end{aligned} \tag{4-8}$$

The averaging time T (or N) in eq. (4-7) must be determined such that further increasing T does not affect $E[f]$ or $E[f^2]$ appreciably. Here T is chosen to be $6600 \Delta h$.

Method II. Numerical Simulation by the Finite-Difference Technique

Instead of solving the reduced single-degree-of-freedom dynamical equation (4-3), the equation of motion (2-8) and the compatibility equation (2-9) are integrated numerically by the finite-difference method employing the simulated white noise $p(\tau)$ as an input. The integration time increment $\Delta \tau$

and the averaging time T are chosen to be the same as those used in method I. The mean-square response of the deflection is then calculated by use of eqs. (4-7) and (4-8).

Method III. Use of the Fokker-Planck Equation [14]

Since the input $Q(\tau)$ in eq. (4-3) is white noise, this equation can be solved exactly by the Fokker-Planck approach. Writing $y_1 = f$ and $y_2 = \dot{f}$, eq. (4-3) is equivalent to the following pair of first-order equations.

$$\dot{y}_1 = y_2 \quad (4-9)$$

$$\dot{y}_2 = -2\zeta\omega_0 y_2 - \omega_0^2(y_1 + \beta y_1^2 + \gamma y_1^3) + Q(\tau)\alpha$$

The stationary Fokker-Planck equation associated with eq. (4-9) is

$$\frac{\pi G_0 \alpha^2}{4} \frac{\partial^2 p}{\partial y_2^2} - \frac{\partial}{\partial y_1} (y_2 p) + \frac{\partial}{\partial y_2} \{ [2\zeta\omega_0 y_2 + \omega_0^2(y_1 + \beta y_1^2 + \gamma y_1^3)] \} p = 0 \quad (4-10)$$

where p is a joint probability density for y_1 and y_2 . The solution for $p(y_1, y_2)$ obtained by Caughey [14] is

$$\begin{aligned} p(y_1, y_2) &= p(f, \dot{f}) \\ &= C \exp \left\{ -\frac{2\zeta\omega_0^3}{\pi\alpha^2 G_0} \left[\frac{1}{2} f^2 + \frac{1}{3} \beta f^3 + \frac{1}{4} \gamma f^4 + \frac{1}{2} \left(\frac{\dot{f}}{\omega_0} \right)^2 \right] \right\} \quad (4-11) \end{aligned}$$

where C is a normalization factor defined by

$$C = 1 / \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(f, \dot{f}) df d\dot{f} \right] \quad (4-12)$$

Since f and \dot{f} are statistically independent,

$$p(f, \dot{f}) = p(f)p(\dot{f})$$

where $p(f)$ is expressed by

$$p(f) = \frac{\exp(-c_1 f^2 - c_2 f^3 - c_3 f^4)}{\int_{-\infty}^{\infty} \exp(-c_1 f^2 - c_2 f^3 - c_3 f^4) df} \quad (4-13)$$

where

$$\begin{aligned} c_1 &= 1/(2\sigma_{f_0}^2) \\ c_2 &= \beta/(4\sigma_{f_0}^2) \\ c_3 &= \gamma/(4\sigma_{f_0}^2) \\ \sigma_{f_0}^2 &= \pi G_0 \alpha^2 / (4\zeta \omega_0^3) \end{aligned} \quad (4-14)$$

and $\sigma_{f_0}^2$ is the mean-square response of the linear system ($\beta=\gamma=0$) corresponding to eq. (4-3). The variance σ_f^2 of f is obtained by

$$\begin{aligned} \sigma_f^2 &= E[f^2] - (E[f])^2 \\ &= \int_{-\infty}^{\infty} f^2 p(f) df - \left[\int_{-\infty}^{\infty} f p(f) df \right]^2 \end{aligned} \quad (4-15)$$

For plates, i.e., $\beta=0$ and $E[f]=0$, σ_f^2 can be expressed by a parabolic cylinder function $D_v(z)$ as [16,28]

$$\begin{aligned} \sigma_f^2 &= E[f^2] \\ &= \int_{-\infty}^{\infty} f^2 p(f) df \\ &= \frac{\int_{-\infty}^{\infty} f^2 \exp(-c_1 f^2 - c_3 f^4) df}{\int_{-\infty}^{\infty} \exp(-c_1 f^2 - c_3 f^4) df} \end{aligned}$$

Introducing the change of variable $f^2 = g$, we find:

$$\begin{aligned}
 &= \frac{\int_{-\infty}^{\infty} g^{1/2} \exp(-c_1 g - c_3 g^2) dg}{\int_{-\infty}^{\infty} \exp(-c_1 g - c_3 g^2) dg} \\
 &= \frac{1}{2(2c_3)^{1/2}} \frac{D_{3/2} \left(\frac{c_1}{(2c_3)^{1/2}} \right)}{D_{1/2} \left(\frac{c_1}{(2c_3)^{1/2}} \right)} \tag{4-16}
 \end{aligned}$$

In the present study the integrals of eq. (4-15) are evaluated numerically.

Method IV. Use of the Equivalent Linearization Technique.

Let us assume that the nonlinear differential equation (4-3) is linearized as

$$\ddot{f} + 2\zeta\omega_0 f + \omega_{oe}^2 f = \alpha Q(\tau) \tag{4-17}$$

where ω_{oe}^2 is the equivalent linear stiffness. The error e caused by this linearization is the difference between eq. (4-3) and eq. (4-17), i.e.,

$$e = (\omega_{oe}^2 - \omega_0^2)f - \omega_0^2(\beta f^2 + \gamma f^3) \tag{4-18}$$

In order to determine ω_{oe}^2 , we choose ω_{oe}^2 so as to minimize the mean-square value of the error e . The mean-square of e is

$$\begin{aligned}
 E[e^2] &= (\omega_{oe}^2 - \omega_0^2)^2 E[f^2] + \omega_0^4 \beta^2 E[f^4] + \omega_0^4 \gamma^2 E[f^6] \\
 &\quad - 2(\omega_{oe}^2 - \omega_0^2) \omega_0^2 \beta E[f^3] + 2\omega_0^4 \gamma \beta E[f^5] - 2\omega_0^2 \gamma (\omega_{oe}^2 - \omega_0^2) E[f^4]
 \end{aligned}$$

Minimization of $E[e^2]$ is achieved by requiring that

$$\frac{dE[e^2]}{d(\omega_{oe}^2)} = 0 \tag{4-19}$$

from which we have

$$\omega_{oe}^2 = \omega_0^2 \left\{ 1 + \frac{\beta E[f^3] + \gamma E[f^4]}{E[f^2]} \right\} \quad (4-20)$$

Since $Q(\tau)$ is a stationary Gaussian random process with zero mean value, the response of the linearized system is also assumed to be stationary Gaussian if the nonlinearity of the system is small. Then, $E[f^4]$ and $E[f^3]$ are evaluated as

$$\begin{aligned} E[f^4] &= 3(E[f^2])^2 \\ E[f^3] &= 0 \end{aligned} \quad (4-21)$$

The mean-square response of the linearized system is found from

$$\begin{aligned} E[f^2] &= \frac{\alpha^2 G_0}{(\omega_{oe}^2 - \omega_0^2)^2 + 4\zeta^2 \omega_0^2 \omega^2} d\omega \\ &= \frac{\pi \alpha^2 G_0}{4\zeta \omega_0^2 \omega_{oe}^2} \end{aligned} \quad (4-22)$$

Substituting eqs. (4-21) and (4-22) into eq. (4-20), we obtain

$$3\gamma(E[f^2]^2 + E[f^2] - \sigma_{f0}^2) = 0 \quad (4-23)$$

where σ_{f0}^2 is the mean-square response of the corresponding linear system of eq. (4-3) and obtained by replacing ω_{oe}^2 by ω_0^2 . Solving for $E[f^2]$ from eq. (4-23), we find

$$E[f^2] = \frac{1}{6\gamma} \left[(1 + 12\gamma\sigma_{f0}^2)^{1/2} - 1 \right] \quad (4-24)$$

Note that if β is not zero, $E[f^3]$ is not zero. However, the estimation of $E[f^3]$ from the linearized system leads to the conclusion that $E[f^3]$ is zero. Therefore, for the case of shells where β is non-zero, this method will not give a good approximate solution.

4.2 Plates

The mean-square deflection response is evaluated by Method I through IV for the following cases:

- (a) Simply Supported Square Plate ($\lambda=1$)
- (b) Simply Supported Rectangular Plate ($\lambda=2$)
- (c) Clamped Square Plate ($\lambda=1$)
- (d) Clamped Rectangular Plate ($\lambda=2$)
- (e) Simply Supported Circular Plate
- (f) Clamped Circular Plate

It is assumed that the supports for all cases do not permit in-plane displacement of the edges of the structure. The governing equations and the boundary conditions are described in Chapter II.

We assume the following form for deflection w :

$$w = f(\tau)\psi(\xi, \eta) = f(\tau)\sin\frac{\pi\xi}{\lambda}\sin\pi\eta \quad \text{for (a) and (b)} \quad (4-25)$$

$$w = f(\tau)\psi(\xi, \eta) = f(\tau)\sin\frac{2\pi\xi}{\lambda}\sin^2\pi\eta \quad \text{for (c) and (d)} \quad (4-26)$$

$$w = f(\tau)\psi(s) = f(\tau)(1-2m_1s^2+m_1m_2s^4) \quad \text{for (e)} \quad (4-27)$$

$$w = f(\tau)\psi(s) = f(\tau)(1-2s^2+s^4) \quad \text{for (f)} \quad (4-28)$$

where $m_1 = (3+v)/(5+v)$ and $m_2 = (1+v)/(3+v)$. All above assumed solutions satisfy the boundary conditions for each case. By substituting w into the

compatibility equation and solving for F , then using Galerkin's method, we have the following final equation for each case:

$$\ddot{f} + 2\zeta\omega_0 \dot{f} + \omega_0^2(f + \gamma f^3) = \alpha Q(\tau)$$

where ω_0^2 , γ and α for each case are listed in Table 4-1. Once ω_0^2 , γ and α are determined, the mean-square response of the deflection can be evaluated by methods I, III, and IV. The damping coefficient is chosen as $\zeta = 0.05$ for each case. The mean-square response of the central deflection versus spectral density G_0 are shown in Figs. 4-1 through 4-6.

The results show that the mean-square response by methods I, III, and IV are reasonably close to each other, however the results obtained by method II deviate very much from those found by the other three methods. This is due to the truncation error and the propagation error involved in the process of numerical integration. (Case (b) was not worked by method II.) Since the computation time required by method II is great (for example for case (a), it took 46 minutes to get the single point in Fig. 4-1.), the method II is not recommended. It is also to be noted that the results obtained by the equivalent linearization technique are smaller than those by the Fokker-Planck approach for all cases.

4.3 Shells

In this section, the variance of deflection of the following shallow shells is evaluated by methods I and IV.

(g) Simply Supported Cylindrical Shells ($k_1=0$ and $k_2=5$, aspect ratio $\lambda = 1.0$). Here, w is assumed to take the form of eq. (4-25).

- (h) Clamped Cylindrical Shells ($k_1=0$ and $k_2=5$, aspect ratio $\lambda = 1.0$). Here, w is assumed to take the form of eq. (4-26).
- (i) Clamped Cylindrical Shells ($k_1=0$ and $k_2=5$, aspect ratio $\lambda = 2.0$). Here, w is assumed to take the form of eq. (4-26).
- (j) Simply Supported Spherical Shell ($k=0.5$). Here, w is assumed to take the form of eq. (4.27).
- (k) Clamped Spherical Shell ($k=1.0$). Here, w is assumed to take the form of eq. (4-28).

Following the same procedure described in section 4.2, we have

$$\ddot{f} + 2\zeta\omega_0 \dot{f} + \omega_0^2(f + \beta f^2 + \gamma f^3) = \alpha Q(\tau) \quad (4-3)$$

where the constants ω_0^2 , β , γ and α for each case are listed in Table 4-1.

The variance σ_w^2 of the central deflection versus power spectral density G_0 for each case is plotted in Fig. 4-7 through 4-11. From these figures it can be seen that the result obtained by the numerical simulation method are in good agreement with the exact solution when σ_w^2 is small.

TABLE 4-1

VALUES OF ω_0^2 , β , γ AND α 1. Simply Supported Rectangular Shells (Cases (a), (b) and (g))

ω_0^2	$(\lambda^2+1)^2 \pi^4 / \lambda^4 + 24(1-\nu^2)(k_1+k_2)^2 (\pi^4 - 64) / \pi^4 (\lambda^2+1)^2$ $+ 768(k_1^2 + 2\nu k_1 k_2 \lambda^2 + k_2^2 \lambda^4) / \lambda^4 \pi^4$
β	$\left\{ -8(1-\nu^2) [4(k_1+k_2)/(\lambda^2+1)^2 + (k_1+k_2 \lambda^2)/\lambda^4] - 72[k_1 + \nu \lambda^2 (k_1+k_2) + k_2 \lambda^4] / \lambda^4 \right\} / \omega_0^2$
γ	$\left\{ 3\pi^4 [(1-\nu^2)(\lambda^4+1)/2 + (\lambda^4 + 2\nu \lambda^2 + 1)] / 8\lambda^4 \right\} / \omega_0^2$
α	$192(1-\nu^2) / \pi^2$

2. Clamped Rectangular Shells (Cases (c), (d), (h) and (i))

ω_0^2	$16[\pi^4(3\lambda^4 + 2\lambda^2 + 3) + 3(k_1^2 + 2\nu \lambda^2 k_1 k_2 + \lambda^4 k_2^2)] / 9\lambda^4 + (1-\nu^2) \times [32(k_1^2 + k_2^2 \lambda^4) / \lambda^4 + (k_1+k_2)^2 / (\lambda^2+1)^2] / 12$
β	$\left\{ -4(1-\nu^2)\pi^2 \left[(k_1+k_2 \lambda^2)/\lambda^4 + (k_1+k_2)/(\lambda^2+1)^2 \right] - 6\pi^2 \times [k_1^2 + \nu \lambda^2 (k_1+k_2) + k_2 \lambda^4] / \lambda^4 \right\} / \omega_0^2$
γ	$\left\{ 51\pi^4(1-\nu^2)(\lambda^4+1) / 36\lambda^4 + \frac{2}{3}(1-\nu^2)\pi^4 \left[1/(\lambda^2+4)^2 + 1/(4\lambda^2+1)^2 + 4/(\lambda^2+1)^2 \right] + \frac{3}{2}\pi^4(\lambda^4 + 2\nu \lambda^2 + 1) / \lambda^4 \right\} / \omega_0^2$
α	$64(1-\nu^2) / 3$

3. Simply Supported Spherical Shells (Cases (e) and (j))

ω_0^2	$\left\{ \frac{8}{3}m_2(2m_2-3) + \frac{m_1}{5}(m_2^2 - 5m_2 + 5)(1-\nu^2)k^2 - 6(2m_2-3)(1+\nu) \right. \\ \left. \times [m_2(5-\nu) - 3(3-\nu)]k^2 \right\} m_1 A$
β	$\left\{ \frac{1}{6}(5m_2^3 - 30m_2^2 + 57m_2 - 36)k^2 - \frac{(1+\nu)}{3}(3m_2^2 - 8m_2 + 6) \right. \\ \left. \times [m_2(5-\nu) - 3(3-\nu)]k - \frac{1}{6}(1+\nu)(2m_2-3)[m_2^2(7-\nu) - 4m_2(5-\nu) \right. \\ \left. + 6(3-\nu)]k \right\} m_1^2 A / \omega_0^2$
γ	$\left\{ \frac{4}{7}(m_2^4 - 7m_2^3 + 266m_2^2 - 35m_2 + 14)(1-\nu^2) - \frac{1}{3}(1+\nu)(3m_2^2 - 8m_2 + 6) \right. \\ \left. \times [m_2^2(7-\nu) - 4m_2(5-\nu) + 6(3-\nu)] \right\} m_1^3 A / \omega_0^2$
α	$60(1-\nu^2)(7+\nu)(5+\nu) / (3\nu^2 + 36\nu + 113)$

4. Clamped Spherical Shells (Case (f) and (k))

ω_0^2	$\frac{1}{3}[320 + 8(7-2\nu)(1+\nu)k^2]$
β	$-20(3-\nu)(1+\nu)k / \omega_0^2$
γ	$\frac{40}{21}(23-9\nu)(1+\nu) / \omega_0^2$
α	$20(1-\nu^2)$

where $A = -120(3+\nu)(5+\nu) / (3\nu^2 + 36\nu + 113)$

V

CONCLUSIONS

The large amplitude vibrations of thin elastic plates and shallow shells having various boundary conditions and subjected to random excitation were investigated by using various approximate techniques. As a preliminary study, the analysis of the response of thin elastic plates and shallow shells with damping undergoing moderately large deflections and subjected to step loading was carried out by the finite-difference method in Chapter II. The steady state response for each case was compared with the existing solution obtained by static analysis to check the accuracy of the approximation. The integration time increment bringing about numerical stability and convergence of the solution for a fixed grid spacing was also determined.

In Chapter III a digital simulation technique for representing a white stationary Gaussian random process was presented. The random vibrations of thin elastic rectangular plates and circular plates subject to white random excitation were simulated numerically by two different methods. The first method consists of three steps: The first step requires the reduction of the governing partial differential equations to a single-degree-of-freedom dynamical equation. This was achieved by assuming the normal deflection to be represented by a product of the generalized coordinates and a coordinate to simulate the random load numerically by the procedure described in Chapter III. Then the desired statistical properties were computed by numerically integrating the dynamical equation by the Runge-Kutta method using the simulated random load as an input.

The second method is to integrate the governing equation of motion and the compatibility equation numerically by the finite-difference methods described in Chapter II again employing the simulated random load as an input. In the second numerical simulation method the total response may contain the response associated with higher modes, however, this method requires much more computing time than the first technique. To compare the results obtained by the two numerical simulation methods, the mean-square response of deflection was determined by both the equivalent linearization technique and by the Fokker-Planck approach. For the cases of rectangular plates and circular plates, the mean-square response of the nonlinear system was found to depart more from the linear response for simply supported plates than for clamped ones. The solutions obtained by the first numerical simulation method were reasonably close to those obtained by the equivalent linearization technique and by the Fokker-Planck approach. However, the second numerical simulation method gave rather poor solutions because of the truncation error and the propagation error involved in the integration process. It can be concluded that considering the accuracy of the solution and the computing time, the second numerical simulation method is not suitable for the simulation of random vibrations of plates unless a much more efficient method for solving the governing partial differential equations is used.

In the last section of Chapter IV, the stationary response of shallow shells to white random excitation was obtained by both the first numerical simulation method and the Fokker-Planck approach, within the framework of a single mode approximation.

NOMENCLATURE

a, b	= dimension of plate or shell
$[A_k], [B_k]$	= square matrices
$[A'_k], [B'_k]$	= square matrices
$[C_k], [D_k]$	= square matrices
c^*, c	= viscous damping and dimensionless viscous damping, respectively
D	= flexural rigidity of plate or shell
E	= Young's modulus
F^*, F	= Airy stress function and dimensionless Airy stress function
h	= thickness of plate or shell
$\{F\}, \{G\}$	= column matrices
$\{L\}, \{K\}$	= column matrices
k_x, k_y, k^*	= curvatures of shell
k_1, k_2, k	= dimensionless curvatures of shell
K_0	= $12(1-\nu^2)$
M_x, M_y	= bending moments
M_r, M_θ	= bending moments
M	= number of interior points plus boundary point in ξ direction
N	= number of interior points plus boundary point in η direction
$P_{i,j}^k, P_i^k$	= points at time level τ_k
Q	= amplitude of lateral load
q^*, q	= lateral load and dimensionless load, respectively

r	= coordinate
r	= ratio $\Delta\xi/\Delta\eta$
R	= base radius of spherical shell
$R(\tau)$	= autocorrelation function
s	= dimensionless coordinate
Δs	= spacing dimension
t	= time
$u(t)$	= unit step function
u^*, v^*	= displacements in x and y directions, respectively
u, v	= dimensionless displacements in ξ and η directions, respectively
U	= column matrix whose elements are W and V
w^*, w	= normal deflection and dimensionless normal deflection, respectively
W	= function defined by eq. (2-19)
V	= function defined by eq. (2-20)
ν	= Poisson's ratio (take $\nu = 0.3$)
τ	= dimensionless time
ρ	= mass density of the shell
λ	= aspect ratio a/b
$\Delta\xi, \Delta\eta$	= spacing dimensions
$\Delta\tau$	= time increment
$\sigma_x^*, \sigma_y^*, \tau_{xy}^*$	= membrane stresses in the middle surface
$\sigma_1, \sigma_2, \tau_{12}$	= dimensionless membrane in the middle surface
$\sigma_r^*, \sigma_\theta^*$	= membrane stresses in the middle surface
σ_s, σ_θ	= dimensionless membrane stresses

B_k	= independent random number
c_1, c_2, c_3	= constants
$D(z)$	= parabolic cylinder function
e	= difference between a nonlinear system and its equivalent linear system
$E[]$	= expectation operator
G_0	= power spectral density of the load
$f(t)$	= generalized coordinate
Δh	= basic time increment
$p(y_1, y_2)$	= a joint probability density for y_1 and y_2
y_1, y_2	= dynamical variables
v_k	= a white Gaussian number
$\alpha, \beta, \gamma, \omega_0$	= constants
ξ	= fraction of critical damping
ω	= frequency
ω_{oe}^2	= equivalent linear stiffness
$\Delta \tau$	= integration time increment
σ_w^2	= variance of the deflection
σ_f^2	= variance of the generalized coordinate f
σ_{fo}^2	= variance of f in a linear system
$\psi(\xi, \eta), \psi(s)$	= coordinate functions

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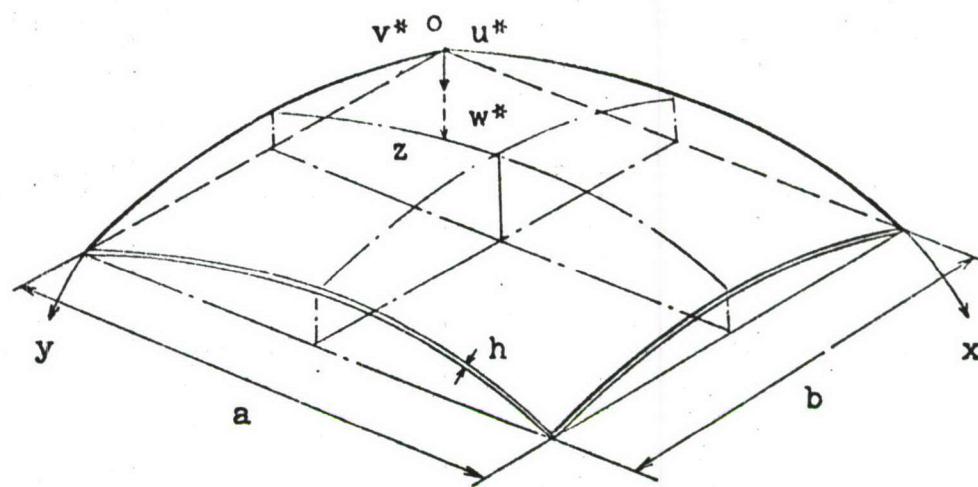


Figure 2-1: Geometry of Shell

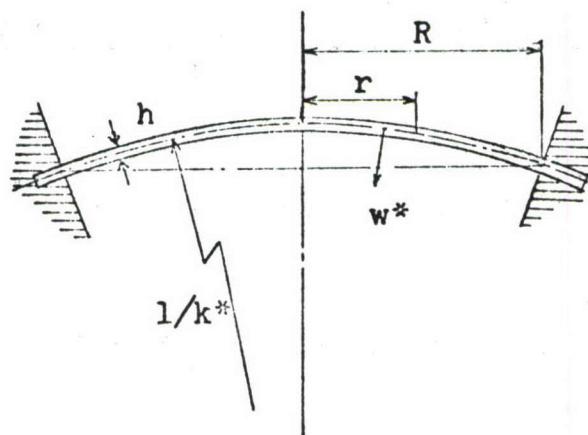


Figure 2-2: Geometry of Shallow Spherical Shell

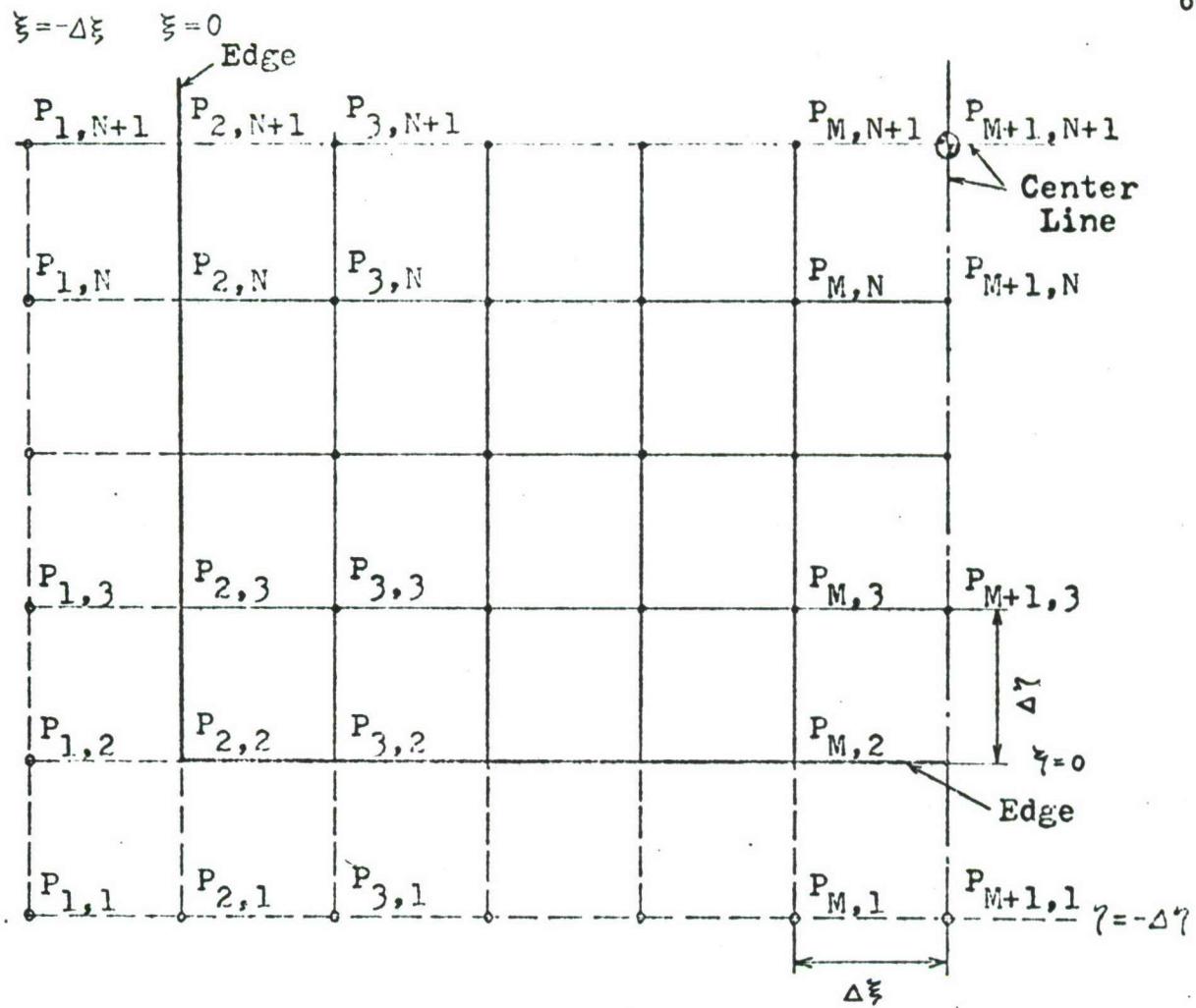
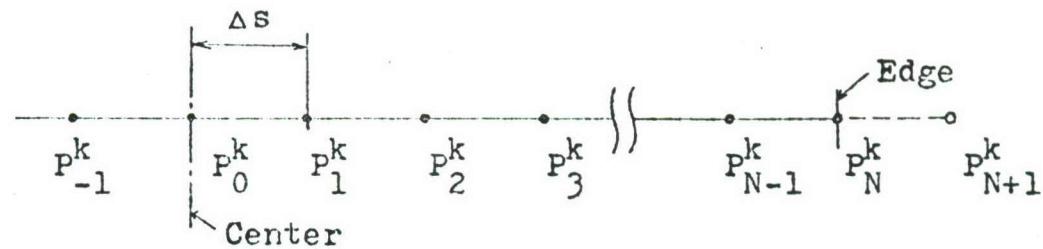


Figure 2-3: Network for Rectangular Plate

Figure 2-4: Descrete Points at Time T_k

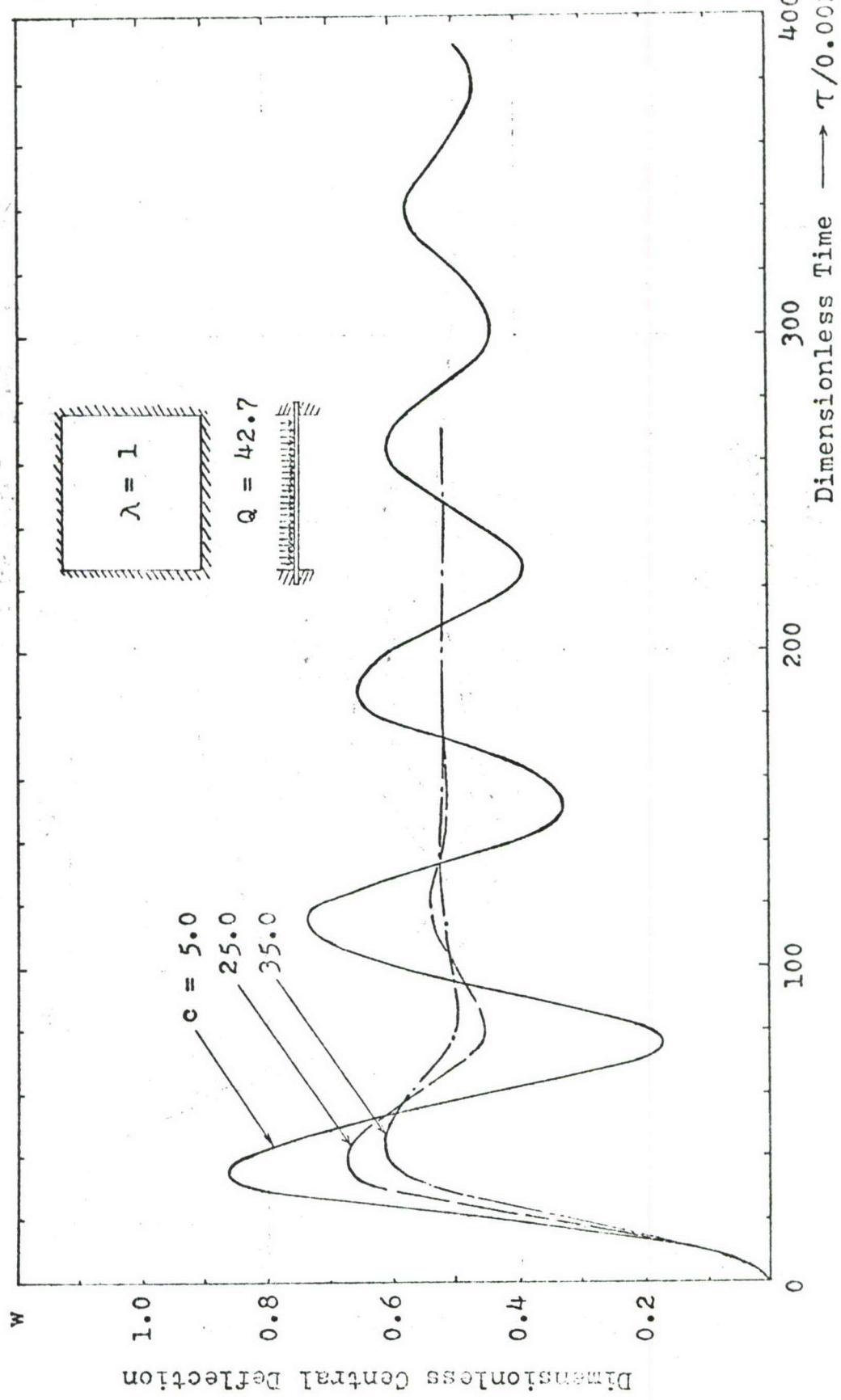


Figure 2-5: Deflection of Clamped Square Plate

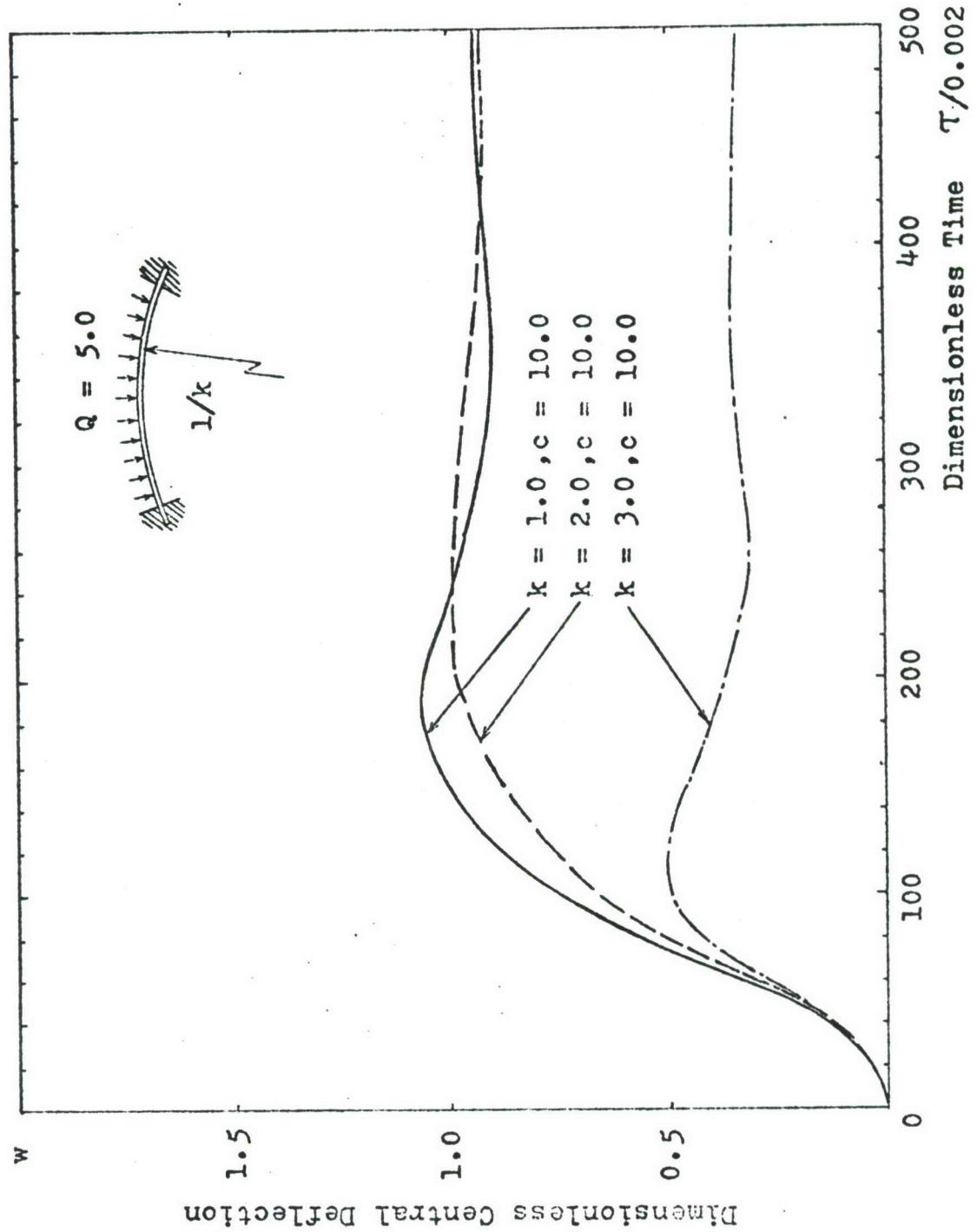


Figure 2-6: Deflection of Clamped Spherical Shell

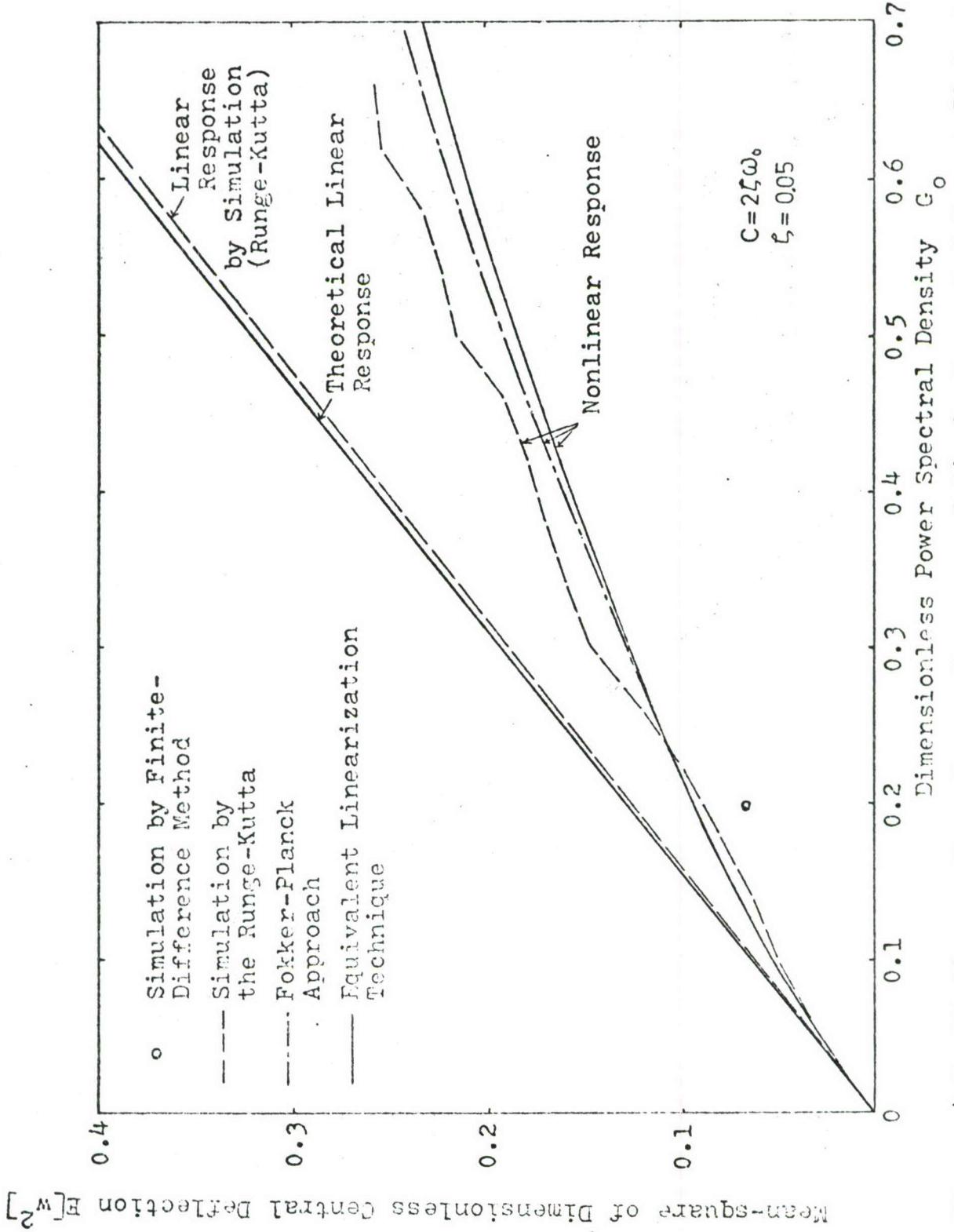


Figure 4-1: Stationary Mean-Square Response of Simply Supported Square Plate to White Noise

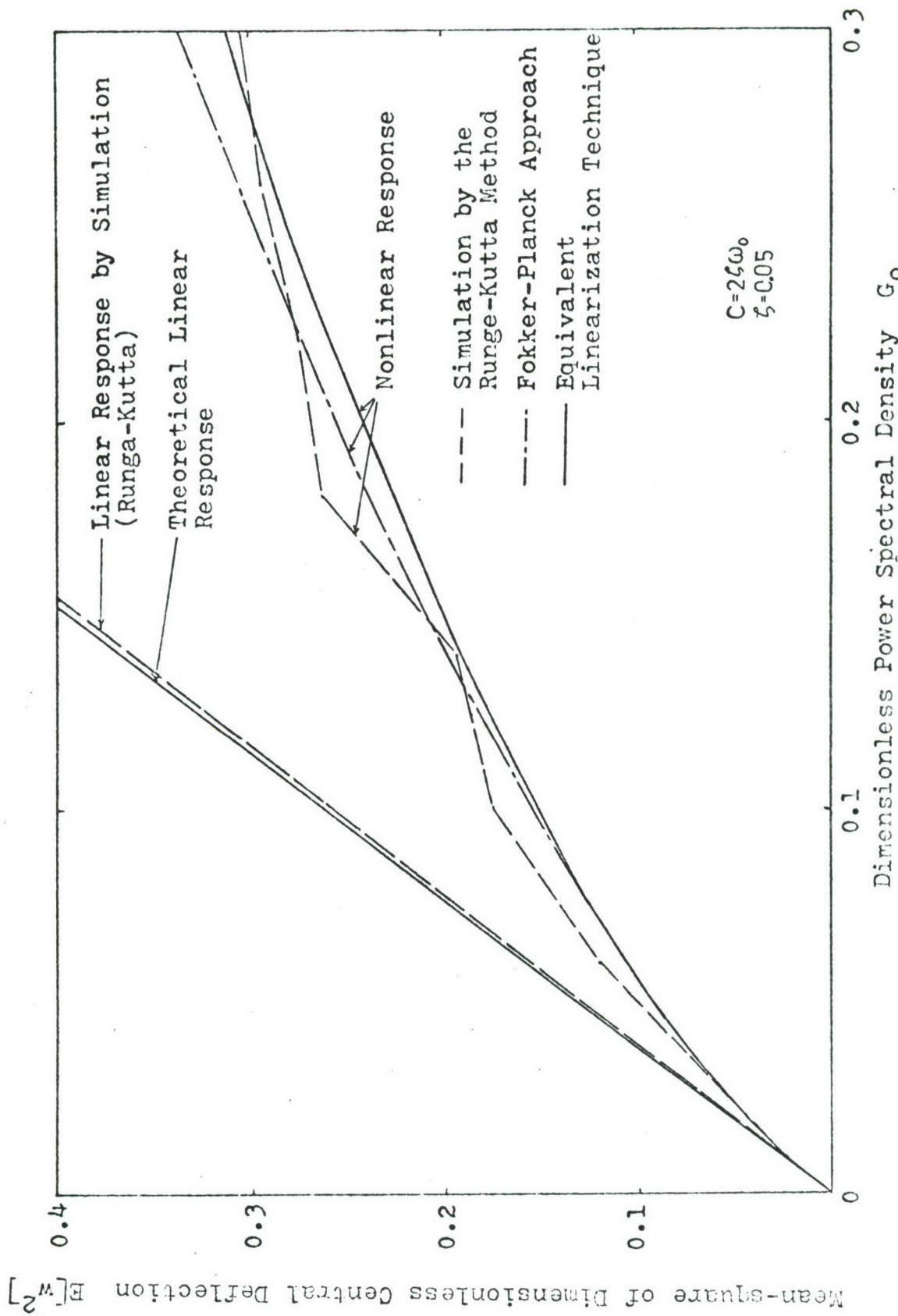


Figure 4-2: Stationary Mean-Square Response of Simply Supported Rectangular Plate (aspect ratio 2) to White Noise

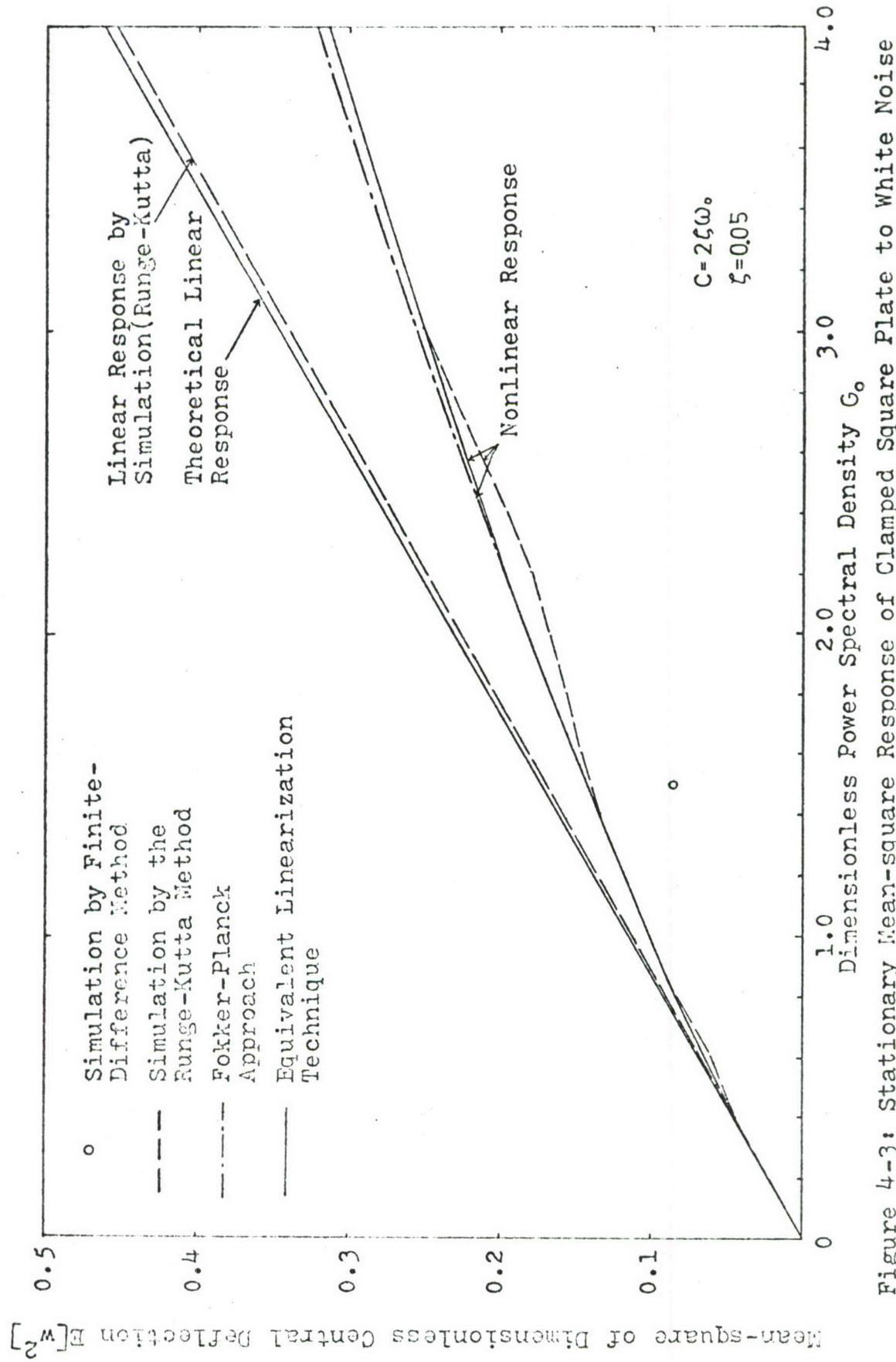


Figure 4-3: Stationary Mean-square Response of Clamped Square Plate to White Noise

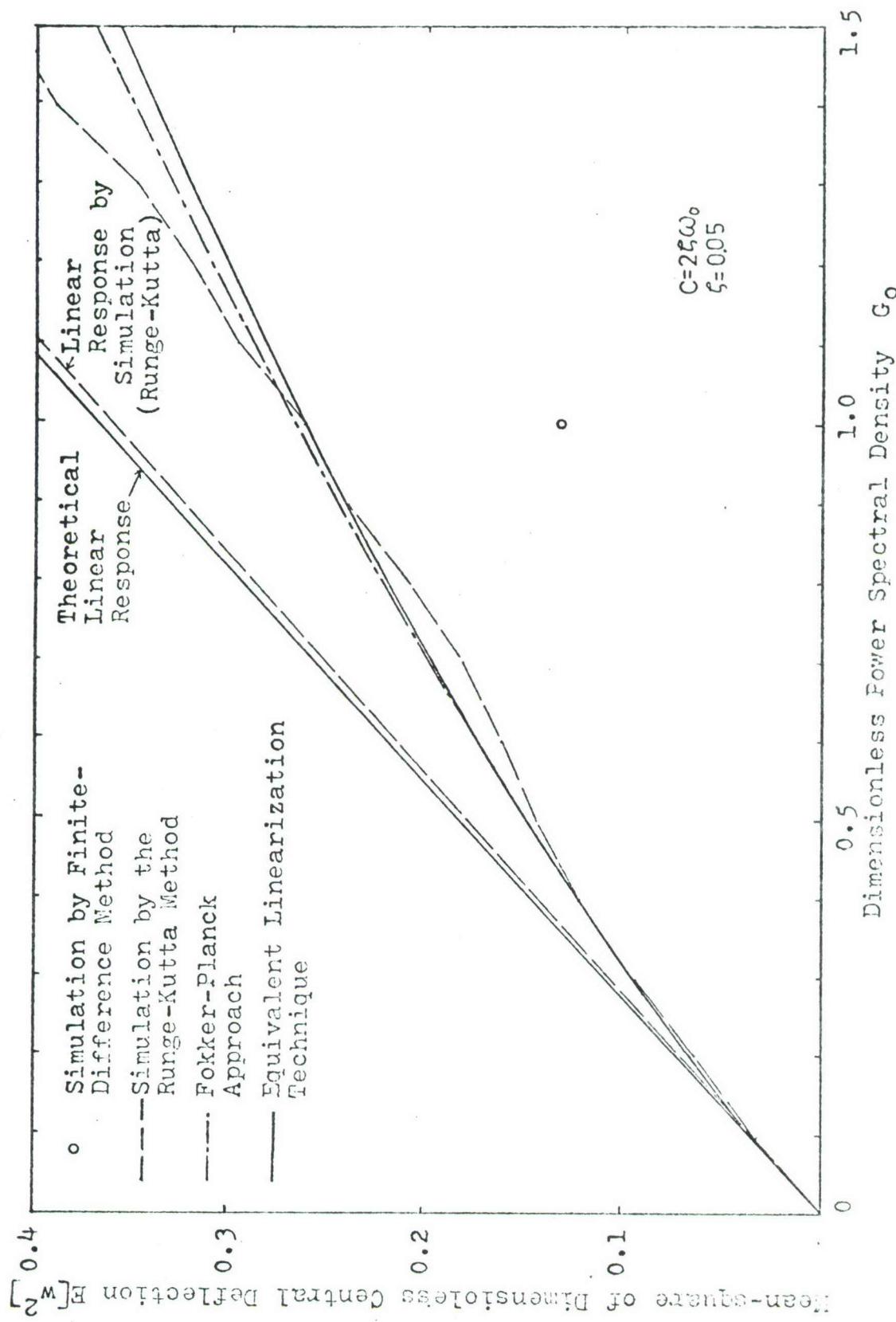


Figure 4-4: Stationary Mean-square Response of Clamped Rectangular Plate (aspect ratio 2) to White Noise

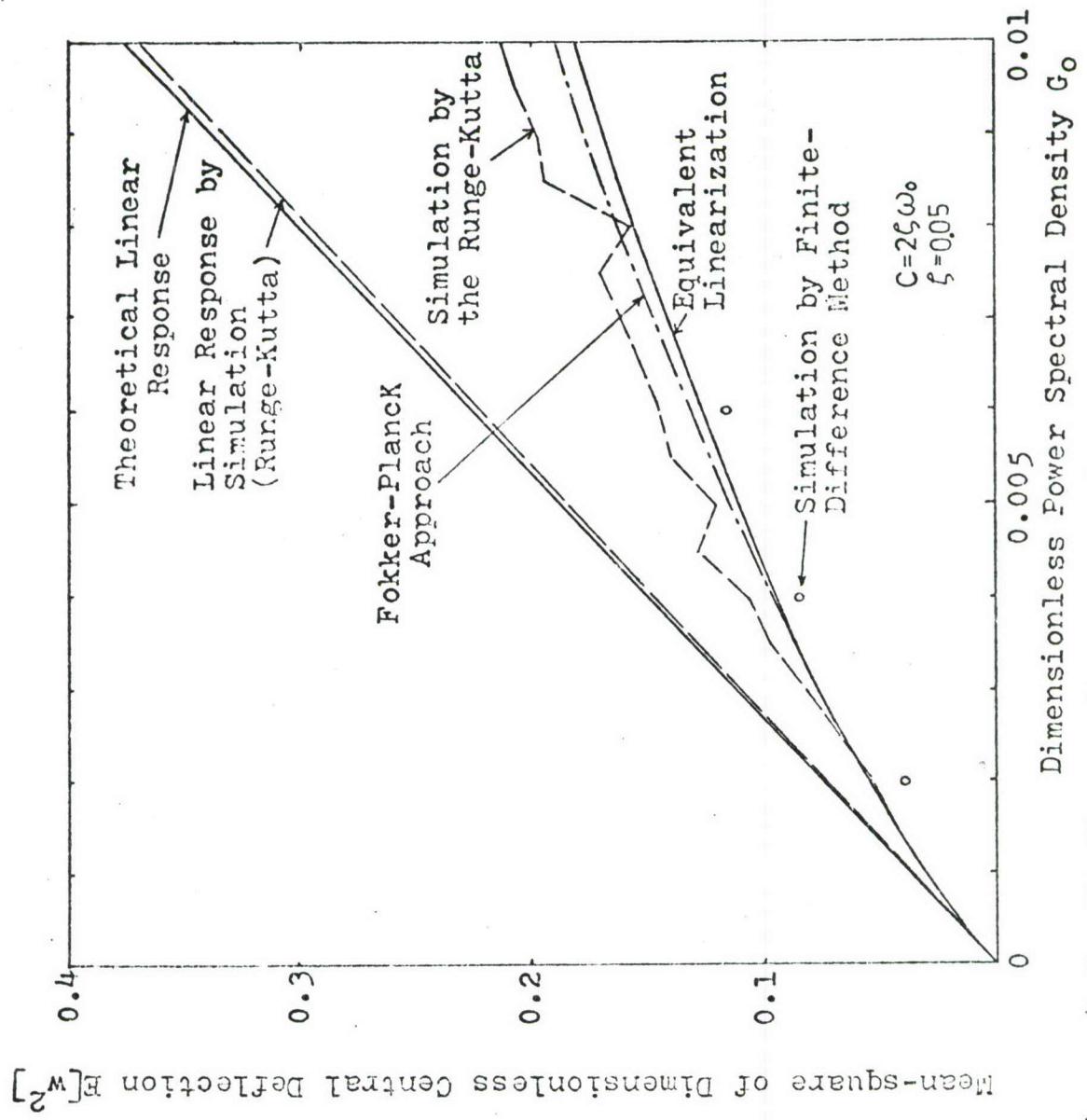
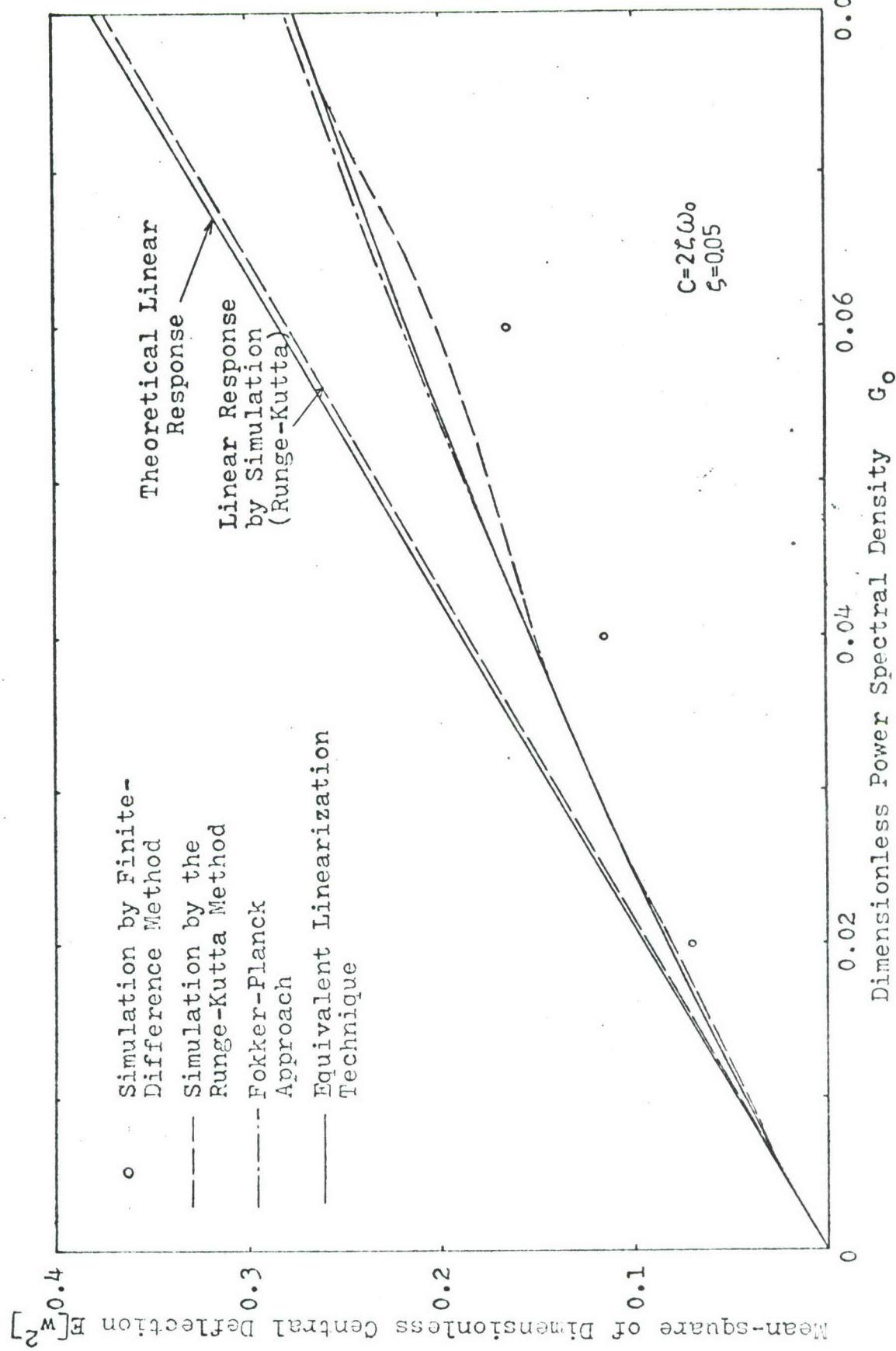


Figure 4-5: Stationary Mean-square Response of Simply Supported Circular Plate to White Noise



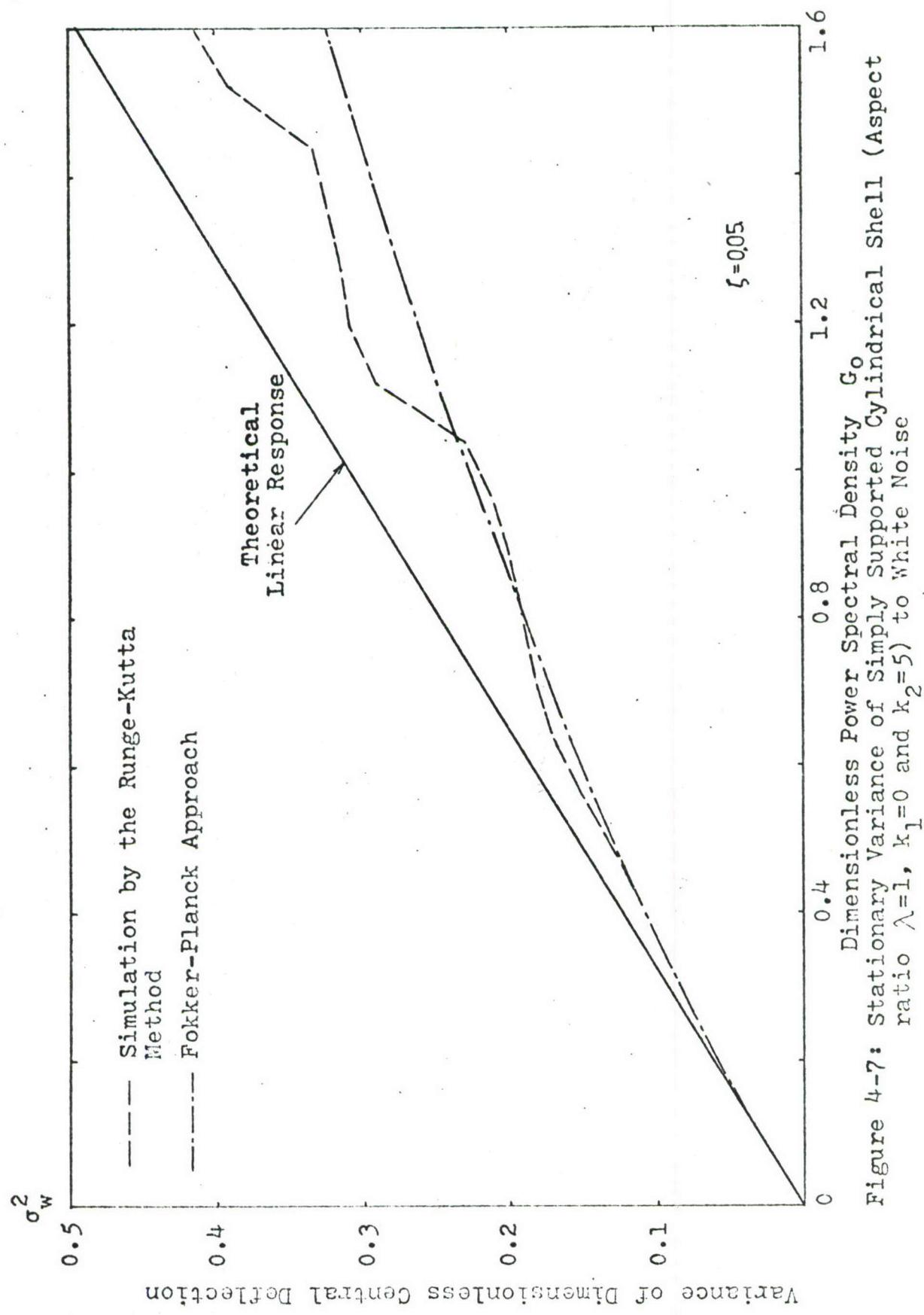


Figure 4-7: Stationary Variance of Simply Supported Cylindrical Shell (Aspect ratio $\lambda=1$, $\kappa_1=0$ and $\kappa_2=5$) to White Noise

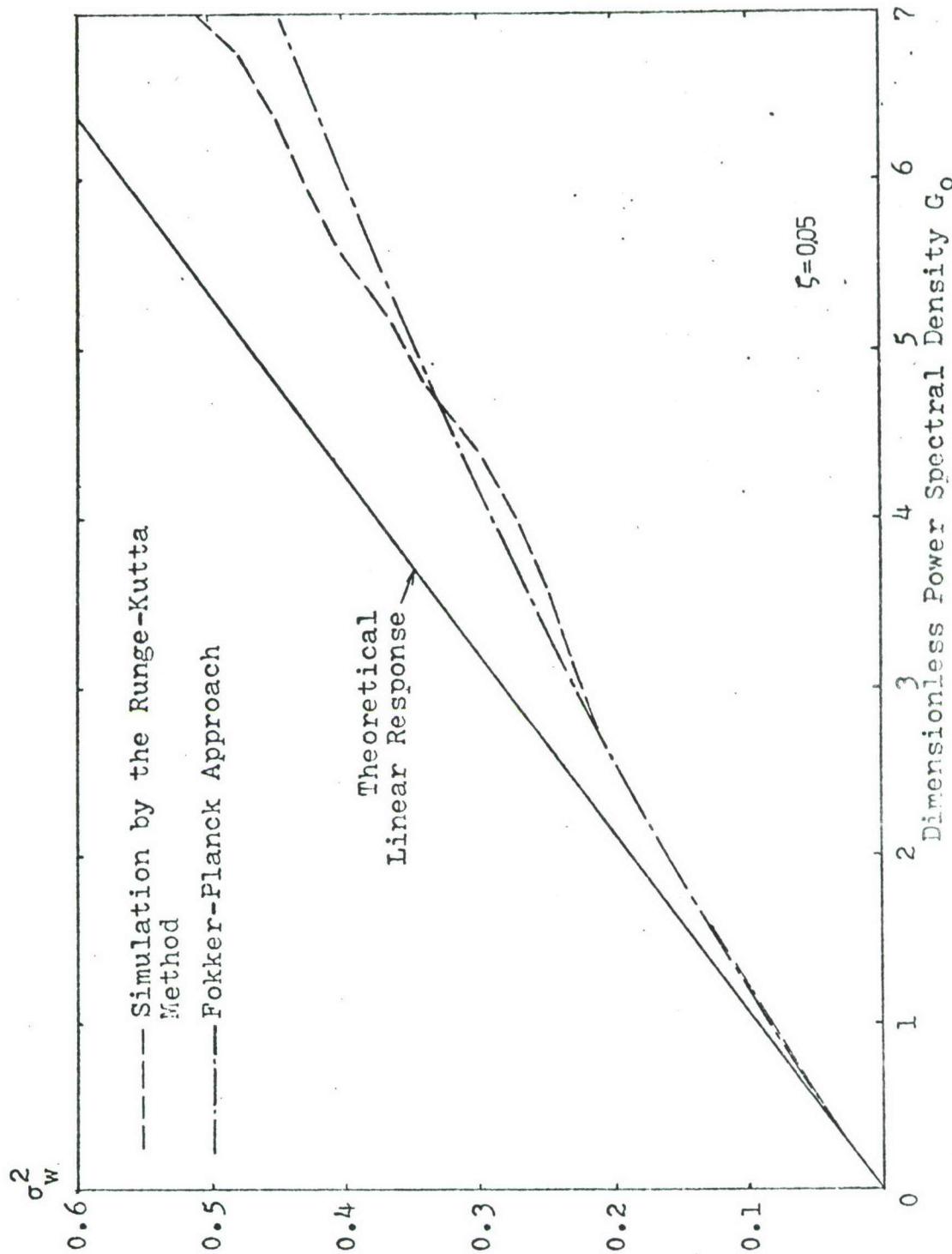


Figure 4-8: Stationary Response of Clamped Cylindrical Shell (Aspect ratio $\lambda=1$, $\kappa_1=0$ and $\kappa_2=5$) to White Noise

Figure 4-8: Stationary Response of Clamped Cylindrical Shell (Aspect ratio $\lambda=1$, $\kappa_1=0$ and $\kappa_2=5$) to White Noise

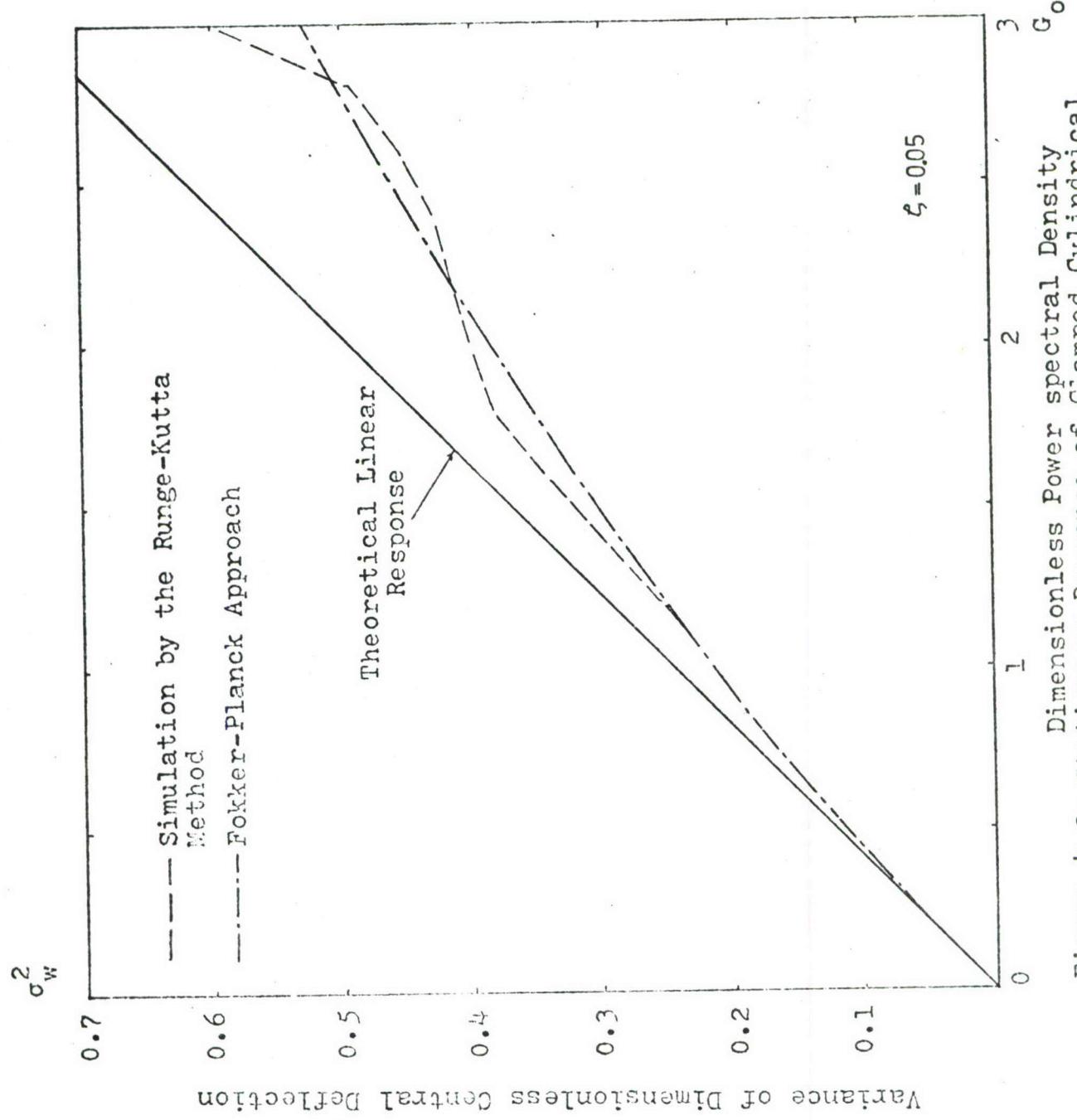


Figure 4-9: Stationary Response of Clamped Cylindrical Shell (Aspect ratio $\lambda=2$, $\kappa_1=0$ and $\kappa_2=5$) to White Noise

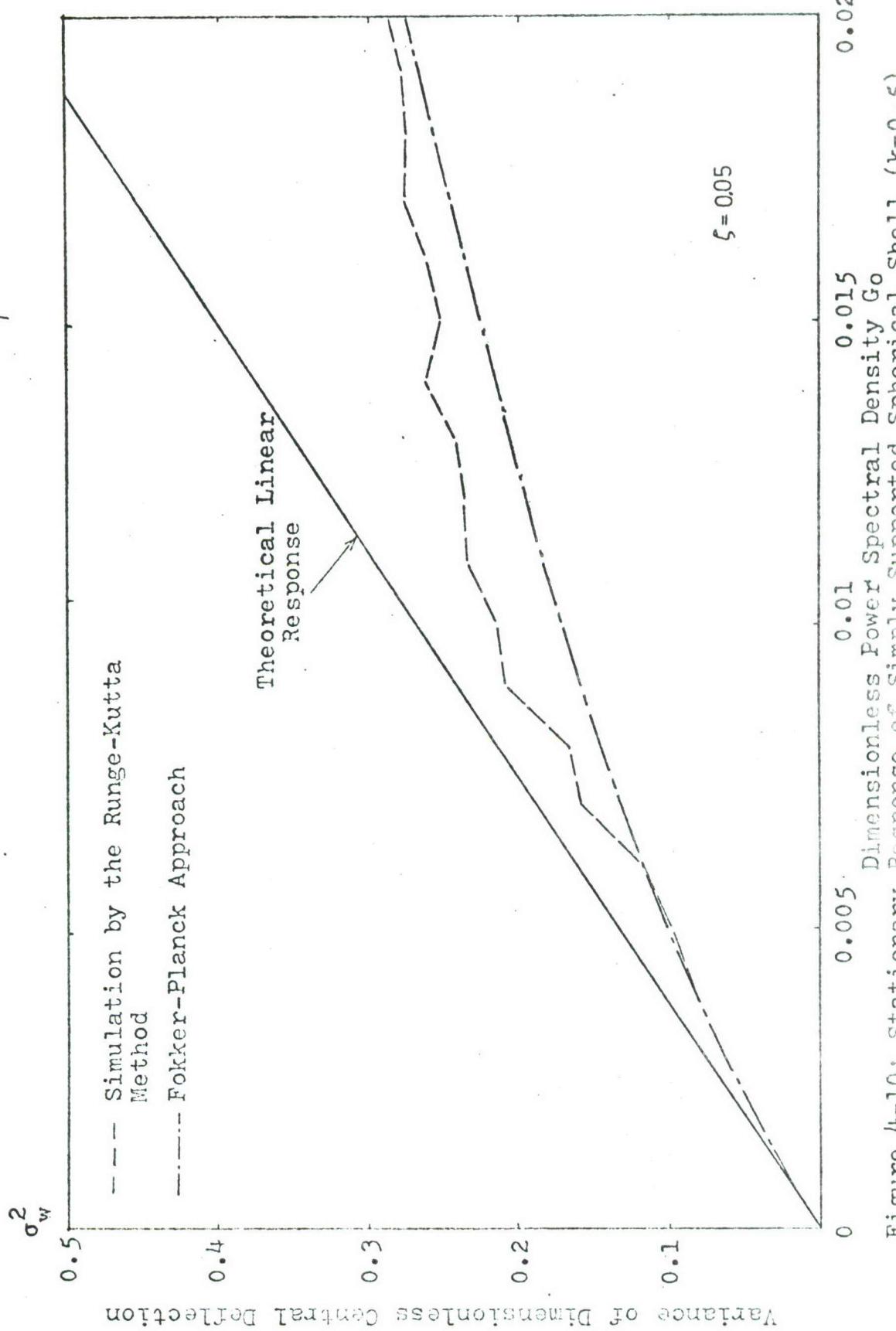
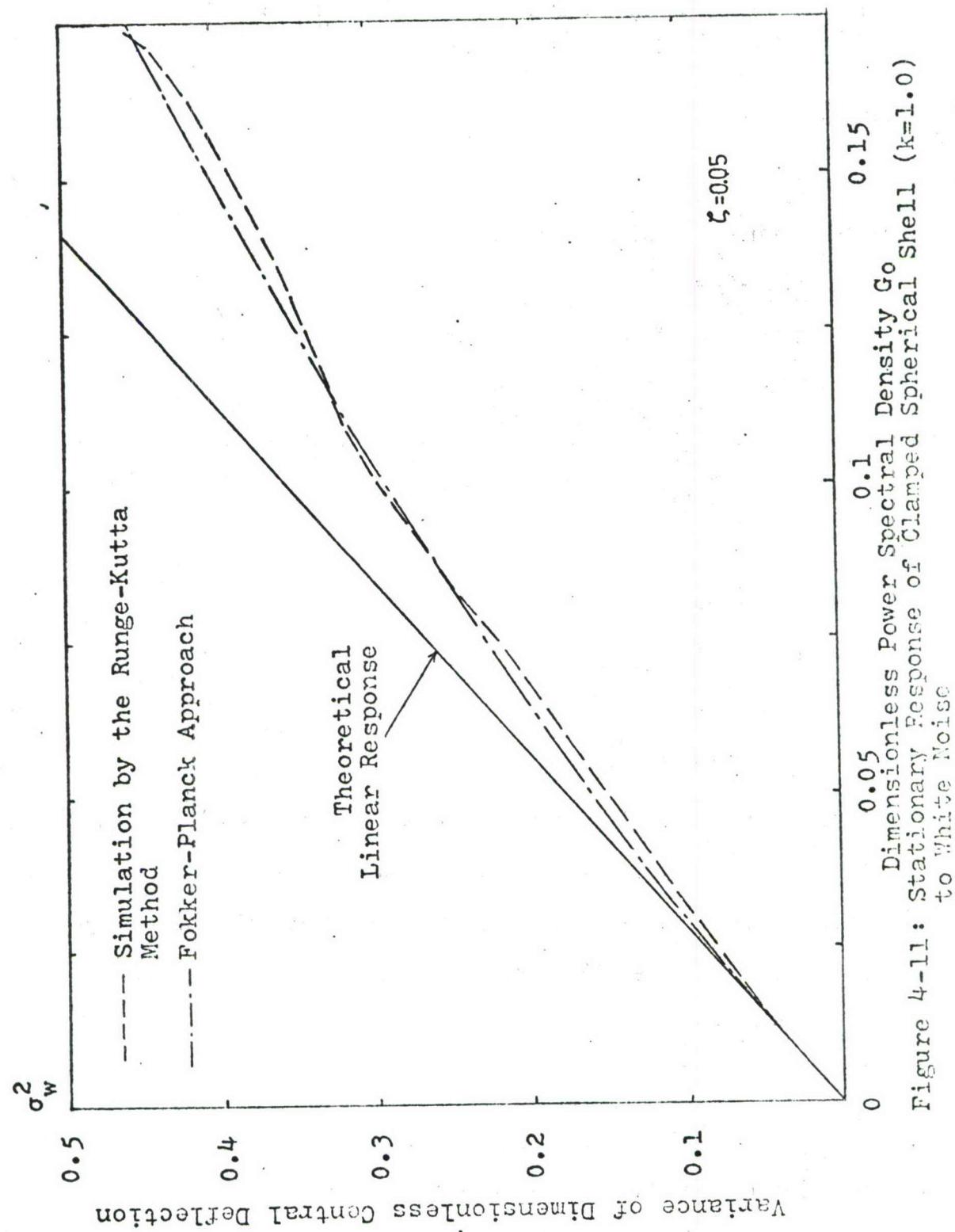


Figure 4-10: Stationary Response of Simply Supported Spherical Shell ($k=0.5$) to White Noise



The Compatibility Equation at the Center of the Shell $s=0$

The compatibility equation in dimensionless form is

$$\frac{\partial^4 F}{\partial s^4} + \frac{2\partial^3 F}{s \partial s^3} - \frac{\partial^2 F}{s^2 \partial s^2} + \frac{\partial F}{s^3 \partial s} = -\frac{\partial^2 w}{\partial s^2} \frac{\partial w}{s \partial s} - k \left(\frac{\partial w}{s \partial s} + \frac{\partial^2 w}{\partial s^2} \right) \quad (A-1)$$

At $s=0$,

$$\frac{\partial w}{\partial s} = 0 \quad (A-2)$$

Therefore, using L'Hospital's rule,

$$\lim_{s \rightarrow 0} \frac{\partial w}{s \partial s} = \frac{\partial^2 w}{\partial s^2} \quad (A-3)$$

The right hand side of eq.(A-1) becomes

$$R.H.S. = -\left(\frac{\partial^2 w}{\partial s^2}\right)^2 - 2k\left(\frac{\partial^2 w}{\partial s^2}\right) \quad (A-4)$$

which is bounded at $s=0$. Expanding F into a Taylor series at $s=0$, we have

$$F = a_0 + a_1 s + a_2 s^2 + \dots + a_n s^n \quad (A-5)$$

where

$$\begin{aligned} a_0 &= F_{s=0} \\ a_1 &= (\partial F / \partial s)_{s=0} \\ &\dots \\ a_n &= \frac{1}{n!} (\partial^n F / \partial s^n)_{s=0} \end{aligned} \quad (A-6)$$

Substituting eq.(A-5) into the left hand side of eq. (A-1), $\nabla^4 F$ is expressed as

$$\nabla^4 F = a_1 s^{-3} + 9a_3 s^{-1} + 64a_4 + 165a_5 s + \dots \quad (A-7)$$

Since the right hand side of eq.(A-1) is bounded at $s=0$, the left hand side must be bounded. From eq.(A-7) we must have

$$a_1 = a_3 = 0 \quad (A-8)$$

Therefore,

$$\lim_{s \rightarrow 0} \nabla^4 F = 64a_4 = \frac{8}{3} \left(\frac{\partial^4 F}{\partial s^4} \right)_{s=0} \quad (A-9)$$

The compatibility equation at $s=0$ is expressed as

$$\frac{8}{3} \frac{\partial^4 F}{\partial s^4} = -\left(\frac{\partial^2 w}{\partial s^2}\right)^2 - 2k\left(\frac{\partial^2 w}{\partial s^2}\right) \quad (A-10)$$

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13. ABSTRACT

The large amplitude vibrations of thin elastic plates and shallow shells having boundary conditions and subjected to random excitation are investigated by using various approximate techniques. The random vibrations of rectangular plates and circular plates subjected to white random excitation are simulated numerically by two different methods. The first method is that the governing equations are reduced to a single-degree-of-freedom dynamical system and the reduced equation is then integrated numerically by the Runge-Kutta method employing the simulated approximate white noise as an input. The second method consists in integrating the equation of motion and the compatibility equation numerically by a finite-difference method employing the simulated approximate white noise as an input. To compare the results obtained by the simulation methods with those by other methods, the single-degree-of-freedom system equation is solved exactly using the Fokker-Planck equation, and solved approximately by the equivalent linearization technique. Also presented is the response analysis of shallow shells to white noise by (1) numerical simulation using the single-degree-of-freedom equation and (2) the Fokker-Planck equation. It has been shown that the solutions by the numerical simulation are close to those obtained by the equivalent linearization technique and the Fokker-Planck approach while the second numerical simulation gives rather poor solutions.

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TECHNIQUES IN SOLVING RANDOM VIBRATION PROBLEMS						

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